7 Multiple integrals

We have finished our discussion of partial derivatives of functions of more than one variable and we move on to integrals of functions of two or three variables.

7.1 Definition of double integral

Consider the function of two variables $f(x, y)$ defined in the bounded region D. Divide the region D into randomly selected n subregions

$$\Delta s_1, \ \Delta s_2, \ \ldots, \ \Delta s_k, \ \ldots, \Delta s_n$$

where Δs_k, $1 \le k \le n$, denotes the kth subregion or the area of this subregion.

Next we choose a random point in every subregion $P_k(\xi_k, \eta_k) \in \Delta s_k$ and multiply the value of the function at the point chosen by the area of the subregion $f(P_k)\Delta s_k$. If we assume that $f(P_k) \ge 0$ then this product equals to the volume of the right prism with the area of base Δs_k and the height $f(P_k)$.

The sum

$$\sum_{k=1}^{n} f(P_k)\Delta s_k$$

is called the *integral sum* of the function $f(x, y)$ over the region D. The geometric meaning is the sum of the volumes of the right prisms, provided $f(x, y) \ge 0$ in the region D.

The maximal distance between the points of the subregion Δs_k is called the *diameter* of this subregion

$$\operatorname{diam} \Delta s_k = \max_{P,Q \in \Delta s_k} |\overrightarrow{PQ}|$$

We have divided the region into subregions randomly. Every subregion has its own diameter. The greatest diameter of subregions we denote by λ, i.e.

$$\lambda = \max_{1 \le k \le n} \operatorname{diam} \Delta s_k$$

Definition 1. If there exists the limit

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} f(P_k)\Delta s_k$$

and this limit does not depend of the choice of subregions of D and the choice of the points P_k in subregions, then this limit is called the *double integral* of the function of two variables $f(x, y)$ over the region D and denoted

$$\iint_D f(x, y)dxdy$$

According to this definition

$$\iint_D f(x, y)dxdy = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(P_k)\Delta s_k \tag{7.1}$$

If $f(x, y) \geq 0$ in the region D then the double integral can be interpreted as the volume of the cylinder between the surface $z = f(x, y)$ and D.

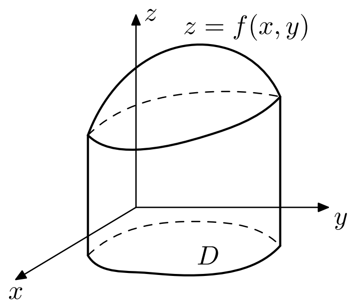

Figure 7.1. The cylinder between the surface $z = f(x, y)$ and D

There holds the following theorem.

Theorem 1. If $f(x, y)$ is continuous in the bounded region D then

$$\iint_D f(x, y)dxdy$$

always exists.

The proof will be omitted. This theorem tells us that for the continuous function $f(x, y)$ the limit

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} f(P_k)\Delta s_k$$

exists and does not depend of the choice of subregions of D and the choice of the points P_k in subregions.

7.2 Properties of double integral

All of the following properties are really just extensions of properties of single integrals.

Property 1. The double integral of the sum of two functions equals to the sum of double integrals of these functions

$$\iint\limits_D [f(x,y)+g(x,y)]dxdy = \iint\limits_D f(x,y)dxdy + \iint\limits_D g(x,y)dxdy$$

provided all three double integrals exist.

Proof. We use the properties of the sum and the limit. By Definition 1

$$\iint\limits_D [f(x,y)+g(x,y)]dxdy = \lim_{\lambda\to 0}\sum_{k=1}^{n}[f(P_k)+g(P_k)]\Delta s_k$$

$$= \lim_{\lambda\to 0}\left[\sum_{k=1}^{n} f(P_k)\Delta s_k + \sum_{k=1}^{n} g(P_k)\Delta s_k\right]$$

$$= \lim_{\lambda\to 0}\sum_{k=1}^{n} f(P_k)\Delta s_k + \lim_{\lambda\to 0}\sum_{k=1}^{n} g(P_k)\Delta s_k.$$

By Definition 1 the first limit equals to $\iint\limits_D f(x,y)dxdy$ and the second limit equals to $\iint\limits_D g(x,y)dxdy$.

Property 2. If c is a constant then

$$\iint\limits_D cf(x,y)dxdy = c\iint\limits_D f(x,y)dxdy$$

i.e. the constant factor can be carried outside the sign of the double integral.

The proof is similar to the proof of Property 1.

Property 3. The double integral of the difference of two functions equals to the difference of double integrals of these functions

$$\iint\limits_D [f(x,y)-g(x,y)]dxdy = \iint\limits_D f(x,y)dxdy - \iint\limits_D g(x,y)dxdy.$$

Property 3 is the conclusion of the properties 1 and 2 because

$$f(x,y)-g(x,y) = f(x,y)+(-1)g(x,y)$$

Property 4. If $D = D_1 \cup D_2$ and the regions D_1 and D_2 have not common interior points then

$$\iint\limits_{D} f(x,y)dxdy = \iint\limits_{D_1} f(x,y)dxdy + \iint\limits_{D_2} f(x,y)dxdy$$

Proof. In the definition of the double integral the limit doesn't depend on the division of the region D. Therefore, starting the random division of the region D we first divide D into D_1 and D_2. Further random division of the region D creates the random divisions into subregions of the regions D_1 and D_2. The integral sum we split into two addends

$$\sum_{k=1}^{n} f(P_k)\Delta s_k = \sum_{D_1} f(P_k)\Delta s_k + \sum_{D_2} f(P_k)\Delta s_k \tag{7.2}$$

where the first addend contains the products, having as one factor the area of the subregions of the region D_1 and the second addend contains the products, having as one factor the area of the subregions of the region D_2, The first sum on the right side of this equality is the integral sum of the function $f(x,y)$ over the region D_1 and the second over the region D_2.

If λ denotes the greatest diameter of the subregions of the region D then $\lambda \to 0$ yields that the greatest diameter of the subregions of D_1 and D_2 approach to zero. We get the assertion of our property if we find the limits of both sides of the equality (7.2) as $\lambda \to 0$.

7.3 Iterated integral. Evaluation of double integral

In the previous subsection we have defined the double integral. However, just like with the definition of a definite integral the definition is very difficult to use in practice and so we need to start looking into how we actually compute double integrals. In this subsection we assume that the bounded region D is closed. There are two types of regions that we need to look at.

The region D is called *regular with respect to y axis* if any straight line parallel to y axis passing the interior points of the region cuts the boundary at two points.

The regular with respect to y axis region can be described by two pairs of inequalities $a \leq x \leq b$ and $\varphi_1(x) \leq y \leq \varphi_2(x)$. This notation is a way of saying we are going to use all the points, (x,y), in which both of the coordinates satisfy the given inequalities.

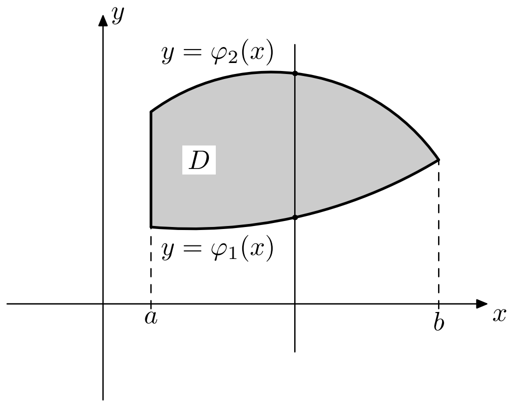

Figure 7.2. Regular region with respect to y axis

Let the function $f(x, y)$ be defined in the region D. The *iterated integral* of the function $f(x, y)$ over this region is defined as follows

$$I_D = \int\limits_a^b \left(\int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \right) dx$$

To compute the iterated integral we integrate first with respect to y by holding x constant as if this were a definite integral. This is called *inner integral* and the result of this integration is the function of one variable x

$$\Phi(x) = \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy$$

Second we multiply the function obtained by dx and compute the *outer integral*

$$\int\limits_a^b \Phi(x)dx$$

which is another definite integral. So, to compute the iterated integral we have to compute two definite integrals. First we integrate with respect to inner variable y and second with respect to outer variable x. To avoid the parenthesis we shall further write the iterated integral as

$$I_D = \int\limits_a^b dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \tag{7.3}$$

Example 1. Compute the iterated integral

$$I_D = \int_0^1 dx \int_0^{x^2} (x^2 + y)dy$$

The region of integration D is described by the inequalities $0 \le x \le 1$ and $0 \le y \le x^2$. The sketch of this region is in Figure 7.3.

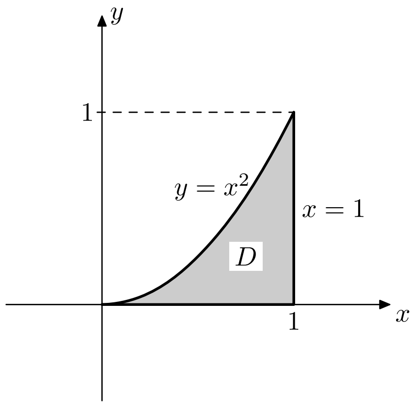

Figure 7.3. The region of integration in Examples 1 and 2

First we compute the inner integral by treating x as constant

$$\Phi(x) = \int_0^{x^2} (x^2 + y)dy = \left(x^2 y + \frac{y^2}{2}\right)\Bigg|_0^{x^2} = x^4 + \frac{x^4}{2} = \frac{3x^4}{2}$$

and then the outer integral

$$I_D = \int_0^1 \frac{3x^4}{2} dx = \frac{3}{2}\frac{x^5}{5}\Bigg|_0^1 = \frac{3}{10}$$

The region D is called *regular with respect to x axis* if any straight line parallel to x axis passing the interior points of the region cuts the boundary at two points.

The regular with respect to x axis region can be described by two pairs of inequalities $c \le y \le d$ and $\psi_1(y) \le x \le \psi_2(y)$.

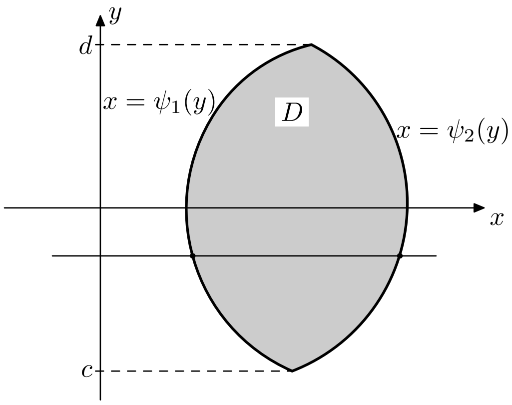

Figure 7.4. Regular region with respect to x-axis

The iterated integral over the region D regular with respect to x axis is defined as

$$I_D = \int\limits_c^d dy \int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx. \tag{7.4}$$

To compute this iterated integral we have to find two definite integrals again. First we integrate with respect to inner variable x by holding y constant. The result is a function of one variable y

$$\Psi(y) = \int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx$$

Second we integrate with respect to outer variable y

$$I_D = \int\limits_c^d \Psi(y)dy$$

In the iterated integral (7.3) the variable y is the inner variable and x is the outer variable, in the iterated integral (7.4) the situation is vice versa. The conversion of the iterated integral from one order of integration to other order of integration is called the *change (or the reverse) of the order of integration.*

Example 2. Change the order of integration in the iterated integral

$$I_D = \int_0^1 dx \int_0^{x^2} f(x,y)dy$$

The region of integration has been sketched in Figure 7.3. In the iterated integral given the inner variable is y and the outer variable x. After changing the order of integration the outer variable has to be y and the inner variable x. Note that the limits of the outer variable are always constants. The limits of the inner variable are in general (but not always) the functions of the outer variable.

Choosing y as the outer variable we can see in Figure 7.3 that the variable y changes between 0 and 1 that is $0 \le y \le 1$. Solving the equation $y = x^2$ for x we get $x = \pm\sqrt{y}$. The domain in this example is bounded by the right branch of the parabola $x = \sqrt{y}$, thus, the variable x in this region is determined by $\sqrt{y} \le x \le 1$. Changing the order of integration, we obtain the iterated integral

$$I_D = \int_0^1 dy \int_{\sqrt{y}}^1 f(x,y)dx$$

Let us compute the iterated integral of Example 1 again, using the reversed order of integration, i.e. compute

$$\int_0^1 dy \int_{\sqrt{y}}^1 (x^2+y)dx$$

Here we integrate first with respect to x

$$\int_{\sqrt{y}}^1 (x^2+y)dx = \left(\frac{x^3}{3} + yx\right)\Big|_{\sqrt{y}}^1 = \frac{1}{3} + y - \frac{y\sqrt{y}}{3} - y\sqrt{y} = \frac{1}{3} + y - \frac{4}{3}y^{\frac{3}{2}}$$

Next we integrate with respect to y

$$\int_0^1 \left(\frac{1}{3} + y - \frac{4}{3}y^{\frac{3}{2}}\right) dy = \left[\frac{y}{3} + \frac{y^2}{2} - \frac{8y^2\sqrt{y}}{15}\right]\Big|_0^1 = \frac{3}{10}$$

The result is, as expected, equal to the result obtained in Example 1.

Example 4. Sometimes we need to change the order of integration to get a tractable integral. For example, if we try to evaluate

$$\int_0^1 dx \int_x^1 e^{y^2}\,dy$$

directly, we shall run into trouble because there is no antiderivative of e^{y^2}, so we get stuck trying to compute the integral with respect to y. But, if we change the order of integration, then we can integrate with respect to x first, which is doable. And, it turns out that the integral with respect to y also becomes possible after we finish integrating with respect to x.

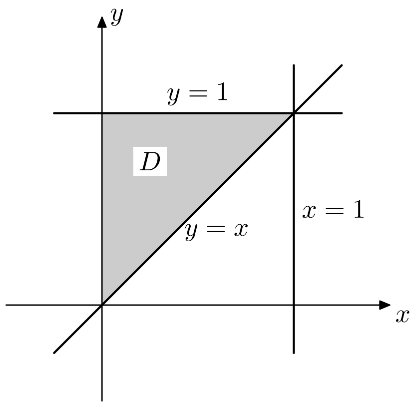

Figure 7.5. The region of Example 3

If we choose the reversed order for integration then the region of integration is described by inequalities $0 \le y \le 1$ and $0 \le x \le y$, thus

$$\int_0^1 dx \int_x^1 e^{y^2}\,dy = \int_0^1 dy \int_0^y e^{y^2}\,dx$$

Evaluating the inner integral with respect to x we treat y as constant, i.e. e^{y^2} is a constant factor and

$$\int_0^y e^{y^2}\,dx = e^{y^2} \cdot x \Big|_0^y = ye^{y^2}$$

Now it is possible to find the outer integral, using the differential $d(y^2) = 2ydy$

$$\int\limits_0^1 ye^{y^2}dy = \frac{1}{2}\int\limits_0^1 e^{y^2}d(y^2) = \frac{1}{2}e^{y^2}\Big|_0^1 = \frac{e-1}{2}$$

Example 4. Change the order of integration in the iterated integral

$$I_D = \int\limits_0^3 dy \int\limits_y^{6-y} f(x,y)dx. \tag{7.5}$$

The region of integration is described by inequalities $0 \le y \le 3$ and $y \le x \le 6-y$. We sketch in the Figure the lines $y = 0$, $y = 3$, $x = y$ and $x = 6 - y$.

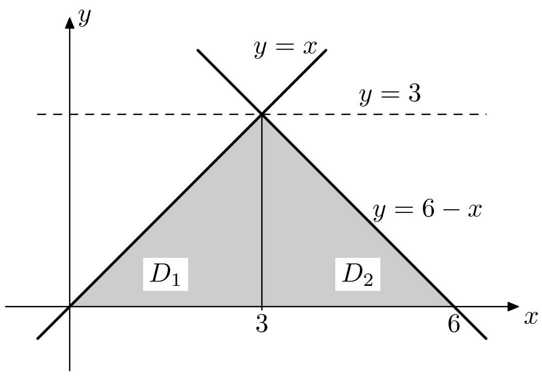

Figure 7.6. Region of integration of Example 4

Obviously in this region $0 \le x \le 6$ and $0 \le y \le \varphi(x)$, where

$$\varphi(x) = \begin{cases} x, & \text{if } 0 \le x \le 3 \\ 6 - x, & \text{if } 3 \le x \le 6 \end{cases}$$

Changing the order of integration

$$I_D = \int\limits_0^6 dx \int\limits_0^{\varphi(x)} f(x,y)dy$$

By the additivity property of the definite integral

$$\begin{aligned} I_D &= \int_0^3 dx \int_0^{\varphi(x)} f(x,y)dy + \int_3^6 dx \int_0^{\varphi(x)} f(x,y)dy \\ &= \int_0^3 dx \int_0^{x} f(x,y)dy + \int_3^6 dx \int_0^{6-x} f(x,y)dy \end{aligned}$$

Dividing the region D by the line $x = 3$ we obtain two regular regions D_1 and D_2. The region D_1 is determined by the inequalities $0 \le x \le 3$ and $0 \le y \le x$ and D_2 by inequalities $3 \le x \le 6$ and $0 \le y \le 6 - x$. The result of the Example 4 we can interpret in two ways. First, if we want to change the order of integration in the iterated integral (7.5), we have to divide the region of integration by the line $x = 3$ into two regions D_1 and D_2 and determine the limits for both regions. Second, if we divide the region of integration D by the line parallel to y axis into two regions D_1 and D_2 so that $D = D_1 \cup D_2$, then

$$I_D = I_{D_1} + I_{D_2}$$

The last assertion holds also if we divide the region by the lines parallel to y axis into three or more regions. It holds as well if we divide the region by the lines parallel to x axis into finite number of subregions. We can conclude the first essential property of the iterated integral.

Property 1. If the regular region D is divided by the lines parallel to coordinate axes into n subregions

$$D = D_1 \cup D_2 \cup \ \ldots \ \cup D_n$$

then

$$I_D = \sum_{k=1}^{n} I_{D_k}$$

We need to prove two more properties of the iterated integral. Suppose the region D is determined by inequalities $a \le x \le b$ and $\varphi_1(x) \le y \le \varphi_2(x)$.

Property 2. If m is the least value and M the greatest value of the function $f(x, y)$ in the regular region D and S_D denotes the area of the region D then

$$m \cdot S_D \le I_D \le M \cdot S_D \tag{7.6}$$

Proof. By the assumption for any $(x, y) \in D$

$$m \le f(x, y) \le M$$

By the property of the definite integral

$$\int\limits_{\varphi_1(x)}^{\varphi_2(x)} m dy \leq \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \leq \int\limits_{\varphi_1(x)}^{\varphi_2(x)} M dy$$

which yields

$$my\Big|_{\varphi_1(x)}^{\varphi_2(x)} \leq \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \leq My\Big|_{\varphi_1(x)}^{\varphi_2(x)}$$

or

$$m[\varphi_2(x) - \varphi_1(x)] \leq \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \leq M[\varphi_2(x) - \varphi_1(x)]$$

Using the same property of the definite integral again, we obtain

$$m\int\limits_a^b [\varphi_2(x) - \varphi_1(x)]dx \leq \int\limits_a^b dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \leq M\int\limits_a^b [\varphi_2(x) - \varphi_1(x)]dx$$

The formula of the area of the region

$$S_D = \int\limits_a^b [\varphi_2(x) - \varphi_1(x)]dx$$

completes the proof.

Property 3. If the function $f(x,y)$ is continuous in the closed regular region D, then there exists $P(\xi,\eta) \in D$ such that

$$I_D = f(P)S_D \tag{7.7}$$

Proof. The continuous in the closed region D function $f(x,y)$ has the least value m and the greatest value M in this region. Hence, there holds the assertion of the Property 2. Dividing the inequalities (7.6) by the area of the region S_D gives

$$m \leq \frac{1}{S_D} I_D \leq M$$

The continuous in the closed region function acquires any value between the least and the greatest, i.e. also the value $\frac{1}{S_D} I_D$. Thus, there exists a point $P(\xi, \eta) \in D$ such that $f(\xi, \eta) = \dfrac{1}{S_D} I_D$. Multiplying the last equation by the area S_D gives us (7.7).

Now it turns out why we have paid so much attention to the iterated integral and what it is useful for.

Theorem. If the function $f(x, y)$ is continuous in the closed regular region D then

$$\iint\limits_D f(x, y)dxdy = I_D \tag{7.8}$$

Proof. If we divide the region D by the lines parallel to coordinate axes into n subregions $D_1,\ D_2,\ \ldots,\ D_n$, then by Property 1 the iterated integral

$$I_D = \sum_{k=1}^{n} I_{D_k}$$

By Property 3 in every subregion D_k there exists a point $P_k(\xi_k, \eta_k) \in D_k$ such that

$$I_{D_k} = f(\xi_k, \eta_k) S_{D_k}$$

Thus,

$$I_D = \sum_{k=1}^{n} f(\xi_k, \eta_k) S_{D_k}$$

Denote by λ the greatest diameter of the subregions D_k $(k = 1,\ 2,\ \ldots,\ n)$. Now we find the limits of both sides of the last equality as $\lambda \to 0$. On the left side there is an iterated integral, which is a constant and the limit is that constant. On the right side there is the integral sum of the function of two variables $f(x, y)$ over the region D. By the assumption the function is continuous, hence, the limit of the integral sum exists and equals to the double integral of the function $f(x, y)$ over the region D.

It has turned out that the iterated integral equals to the double integral and so we don't need the term iterated integral any more. It's just a mean to compute the double integral and usually we shall say instead of iterated integral double integral.

If the region D is regular with respect to y axis then we compute the double integral by the formula

$$\iint\limits_D f(x, y)dxdy = \int\limits_a^b dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \tag{7.9}$$

If the region D is regular with respect to x axis then we compute the double integral by the formula

$$\iint_D f(x,y)dxdy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx. \tag{7.10}$$

If the region is regular with respect to either of the coordinate axes then we can choose one of these formulas to compute the double integral.

Example 5. Compute the double integral $\iint_D (x+y)dxdy$ if D is the region bounded by the line $x+y=2$ and parabola $y=x^2$.

To sketch the region we find the intersection points of the parabola and line solving the system of equations

$$\begin{cases} y = x^2 \\ x + y = 2 \end{cases}$$

The second equation gives $y = 2 - x$. Substituting y to the first equation gives the quadratic equation $x^2 + x - 2 = 0$, whose roots are $x_1 = -2$ and $x_2 = 1$.

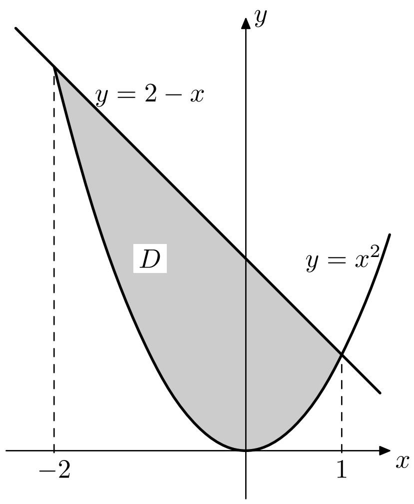

Figure 7.7. The region of integration of Example 5

Using the sketch in Figure 7.7, we determine the limits of integration

$-2 \leq x \leq 1$ and $x^2 \leq y \leq 2-x$. By the formula (7.9)

$$\iint\limits_D (x+y)dxdy = \int\limits_{-2}^{1} dx \int\limits_{x^2}^{2-x} (x+y)dy$$

Compute the inner integral

$$\int\limits_{x^2}^{2-x} (x+y)dy = \left(xy + \frac{y^2}{2}\right)\Bigg|_{x^2}^{2-x}$$

$$= \quad x(2-x) + \frac{(2-x)^2}{2} - x^3 - \frac{x^4}{2} = 2 - \frac{x^2}{2} - x^3 - \frac{x^4}{2}$$

and the outer integral

$$\int\limits_{-2}^{1} \left(2 - \frac{x^2}{2} - x^3 - \frac{x^4}{2}\right) dx = \left(2x - \frac{x^3}{6} - \frac{x^4}{4} - \frac{x^5}{10}\right)\Bigg|_{-2}^{1}$$

$$= \quad 2 - \frac{1}{6} - \frac{1}{4} - \frac{1}{10} - \left(-4 + \frac{4}{3} - 4 + \frac{16}{5}\right) = 4,95.$$

7.4 Change of variable in double integral

Often the reason for changing variables is to get us an integral that we can do with the new variables. Another reason for changing variables is to convert the region into a nicer region to work with. The following example gives an idea why the change of variable can be useful.

Example 1. Compute the double integral

$$\iint\limits_D (2x-3y-4)^2 dxdy$$

where D is the region bounded by the lines $x+y=-1$, $x+y=3$, $3y-2x=6$ and $2x-3y=12$.

Notice that the first two lines are parallel and the third and fourth lines are parallel. The intersection point of the first and third line is $A\left(-\frac{9}{5}; \frac{4}{5}\right)$, the intersection point of the first and fourth line is $B\left(\frac{9}{5}; -\frac{14}{5}\right)$, the intersection point of the second and fourth line is $C\left(\frac{21}{5}; -\frac{6}{5}\right)$ and the intersection point of the second and third line is $D\left(\frac{3}{5}; \frac{12}{5}\right)$.

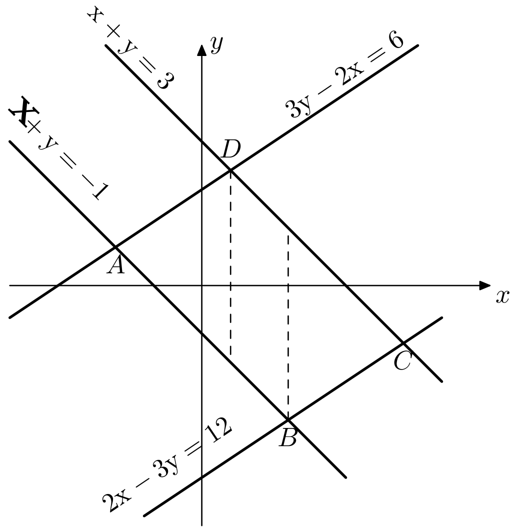

Figure 7.8. The region of integration of Example 1

To compute this double integral by the formula (7.9), we have to divide the region with two vertical lines passing D and B into three subregions, compute this double integral over these three subregions and add the results. The integration demands quite a lot of technical work, which can be avoided if we use the change of variables.

Change the variables x and y by the variables u and v by the equations

$$\begin{cases} x = \varphi(u, v) \\ y = \psi(u, v) \end{cases} \tag{7.11}$$

We assume that $x = \varphi(u, v)$ and $y = \psi(u, v)$ are one-valued continuous functions in the respective region of the uv plane and they have continuous partial derivatives with respect to both variables in that region. In addition we assume that the system of equations (7.11) can be uniquely solved for the variables u and v. Then to any point of the region D in xy plane there is related one point of the region D' in uv plane and vice versa. Divide the region D' in uv plane into subregions with the lines parallel to coordinate axes. Choose the random subregion $\Delta s'$ of the region D' The subregion $\Delta s'$ is a rectangle, whose area is $\Delta s' = \Delta u \Delta v$. On the vertical lines u is constant and the equations (7.11) are the parametric equations of some curve in xy plane and v is the parameter. On the horizontal lines v is constant and the equations (7.11) are also the parametric equations of some curve in xy plane and u is the parameter. Thus, to the vertical lines $u = const$ and $u + \Delta u =$

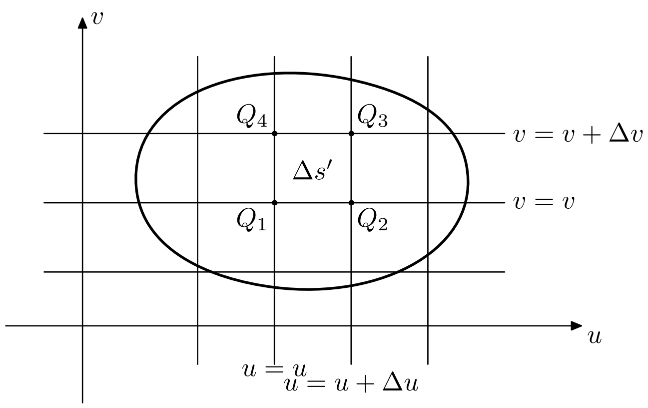

Figure 7.9. The region D'

const there correspond two curves in xy-plane and to the horizontal lines $v =$ *const* and $v + \Delta v =$ *const* there correspond two another curves in xy plane. After that, to the point $Q_1(u, v)$ there is related a point $P_1(\varphi(u, v), \psi(u, v))$, to the point $Q_2(u + \Delta u, v)$ there is related a point $P_2(\varphi(u + \Delta u, v), \psi(u + \Delta u, v))$, to the point $Q_3(u + \Delta u, v + \Delta v)$ there is related a point $P_3(\varphi(u + \Delta u, v + \Delta v), \psi(u + \Delta u, v + \Delta v))$ and to the point $Q_4(u, v + \Delta v)$ there is related a point $P_4(\varphi(u, v + \Delta v), \psi(u, v + \Delta v))$. Therefore, to the rectangle $Q_1Q_2Q_3Q_4$ there is related quadrangle $P_1P_2P_3P_4$, whose sides are curves.

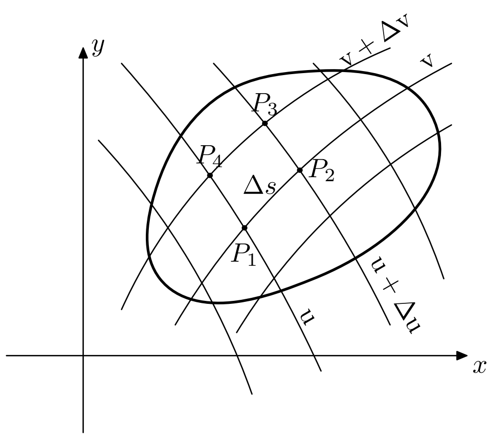

Figure 7.10. The region D

We approximate the quadrangle Δs by the parallelogram built on the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_4}$. The area of the parallelogram built on two vectors equals to the absolute value of the determinant, whose rows are the components of these vectors. Since $\overrightarrow{P_1P_2} = (\varphi(u+\Delta u, v) - \varphi(u,v), \psi(u+\Delta u, v) - \psi(u,v)$ and $\overrightarrow{P_1P_4} = (\varphi(u, v+\Delta v) - \varphi(u,v), \psi(u, v+\Delta v - \psi(u,v))$, then

$$\Delta s = |\begin{vmatrix} \varphi(u+\Delta u, v) - \varphi(u,v) & \psi(u+\Delta u, v) - \psi(u,v) \\ \varphi(u, v+\Delta v) - \varphi(u,v) & \psi(u, v+\Delta v - \psi(u,v) \end{vmatrix}|$$

By the assumptions made the total increments of the functions $x = \varphi(u,v)$ and $y = \psi(u,v)$ can be written

$$\Delta x = \frac{\partial x}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v + \varepsilon_1 \Delta u + \varepsilon_2 \Delta v$$

and

$$\Delta y = \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v + \varepsilon_3 \Delta u + \varepsilon_4 \Delta v$$

where ε_1, ε_2, ε_3 and ε_4 are infinitesimals as $(\Delta u, \Delta v) \to (0;0)$.

In the first row of the determinant of Δs there are the increments of the functions x and y with respect to u and in the second row with respect to v. In the first row $\Delta v = 0$ and in the second row $\Delta u = 0$. Hence,

$$\varphi(u+\Delta u, v) - \varphi(u,v) = \frac{\partial x}{\partial u}\Delta u + \varepsilon_1 \Delta u$$

$$\psi(u+\Delta u, v) - \psi(u,v) = \frac{\partial y}{\partial u}\Delta u + \varepsilon_3 \Delta u$$

$$\varphi(u, v+\Delta v) - \varphi(u,v) = \frac{\partial x}{\partial v}\Delta v + \varepsilon_2 \Delta v$$

and

$$\psi(u, v+\Delta v) - \psi(u,v) = \frac{\partial y}{\partial v}\Delta v + \varepsilon_4 \Delta v$$

Ignoring the infinitesimals of a higher order $\varepsilon_1\Delta u$, $\varepsilon_3\Delta u$, $\varepsilon_2\Delta v$ and $\varepsilon_4\Delta v$ with respect to Δu and Δv, we get the approximate formula to compute the area Δs

$$\Delta s \approx |\begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v \end{vmatrix}| = |\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}|\Delta u \Delta v \tag{7.12}$$

The functional determinant in the formula (7.12) is called *jacobian* and denoted

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{7.13}$$

Now we have the approximate equality between the areas of subregions D and D'

$$\Delta s \approx |J|\Delta s' \tag{7.14}$$

which attains more accuracy as Δu and Δv are getting smaller (since the functions are continuous, Δx and Δy are also getting smaller). The double integral is defined as the limit of the integral sum as the greatest diameter of the subregions approaches zero, i.e. (not writing the indexes)

$$\iint\limits_D f(x,y)dxdy = \lim_{\lambda\to 0} \sum f(\xi,\eta)\Delta s$$

where $P(\xi,\eta)$ is a point in the subregion Δs chosen randomly. Let $(\overline{u},\overline{v})$ be a point in the subregion $\Delta s'$ related to the point $P(\xi,\eta) \in \Delta s$. By our assumptions this is uniquely determined. Using the equality (7.14), we obtain

$$\sum f(\xi,\eta)\Delta s = \sum f(\varphi(\overline{u},\overline{v}),\psi(\overline{u},\overline{v}))|J|\Delta s' \tag{7.15}$$

where the last summation is over all subregions $\Delta s'$ of the region D'. Let λ' denotes the greatest diameter of the subregions of the region D'. Then

$$\lim_{\lambda'\to 0} \sum f(\varphi(\overline{u},\overline{v}),\psi(\overline{u},\overline{v}))|J|\Delta s' = \iint\limits_{D'} f(\varphi(u,v),\psi(u,v))|J|dudv$$

If $\lambda' \to 0$, then the continuity of the functions $x = \varphi(u,v)$ and $y = \psi(u,v)$ yields $\lambda \to 0$. Finding the limit as $\lambda' \to 0$ of both sides of the equality (7.15) gives us the formula of the change of variable in the double integral

$$\iint\limits_D f(x,y)dxdy = \iint\limits_{D'} f(\varphi(u,v),\psi(u,v))|J|dudv \tag{7.16}$$

Now return to Example 1 of this subsection at let's change the variable setting

$$\begin{cases} u = x + y \\ v = 2x - 3y \end{cases} \tag{7.17}$$

This way the parallelogram in xy plane converts to the rectangle in uv plane determined by the inequalities $-1 \le u \le 3$ and $-6 \le v \le 12$.The integrand converts to $(2x-3y-4)^2 = (v-4)^2$. To compute the jacobian (7.13) we solve the system of equations (7.17) for the variables x and y

$$\begin{cases} x = \dfrac{3}{5}u + \dfrac{1}{5}v \\[2ex] y = \dfrac{2}{5}u - \dfrac{1}{5}v \end{cases}$$

and find the partial derivatives

$$\frac{\partial x}{\partial u} = \frac{3}{5}, \quad \frac{\partial y}{\partial u} = \frac{2}{5}$$

$$\frac{\partial x}{\partial v} = \frac{1}{5}, \quad \frac{\partial y}{\partial v} = -\frac{1}{5}$$

Thus, the jacobian

$$J = \begin{vmatrix} \frac{3}{5} & \frac{2}{5} \\ \\ \frac{1}{5} & -\frac{1}{5} \end{vmatrix} = -\frac{3}{25} - \frac{2}{25} = -\frac{1}{5}$$

and by the formula of change of variable (7.16) we find

$$\iint\limits_{D} (2x - 3y - 4)^2 dxdy = \iint\limits_{D'} (v-4)^2 \frac{1}{5} dudv = \frac{1}{5} \int\limits_{-1}^{3} du \int\limits_{-6}^{12} (v-4)^2 dv = 403\frac{1}{5}$$

7.5 Double integral in polar coordinates

If we substitute the Cartesian coordinates x and y by the polar coordinates φ and ρ, we use the formulas

$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \end{cases} \tag{7.18}$$

Recall that φ denotes the polar angle and ρ the polar radius.

To the constant polar angles there correspond the straight lines passing the origin in xy plane and to the constant polar radius there correspond the circles centered at the origin in xy plane. Therefore, first of all we use the change of variable (7.18) if the region of integration is a disk or the part of disk.

To use the formula (7.16) we find $f(x, y) = f(\rho\cos\varphi, \rho\sin\varphi)$ and the jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} \end{vmatrix} = \begin{vmatrix} -\rho\sin\varphi & \rho\cos\varphi \\ \cos\varphi & \sin\varphi \end{vmatrix} = -\rho$$

Since ρ is the polar radius, which is non-negative, we have $|J| = \rho$.

Suppose the region D in xy plane converts to the region Δ in $\varphi\rho$ plane. Then the general formula (7.16) gives us the formula to convert the double integral into polar coordinates

$$\iint\limits_D f(x,y)dxdy = \iint\limits_\Delta f(\rho\cos\varphi, \rho\sin\varphi)\rho d\varphi d\varrho \tag{7.19}$$

Example 1. Convert to polar coordinates the double integral

$$\iint\limits_D f(x,y)dxdy$$

if the region of integration D is the disk $x^2 + y^2 \le 4y$.

The region D is bounded by the circle $x^2 + y^2 = 4y$. Converting this equation, we have $x^2+y^2-4y = 0$, i.e. $x^2+y^2-4y+4 = 4$ or $x^2+(y-2)^2 = 4$. This is the equation of the circle with radius 2 centered at the point $(0;2)$.

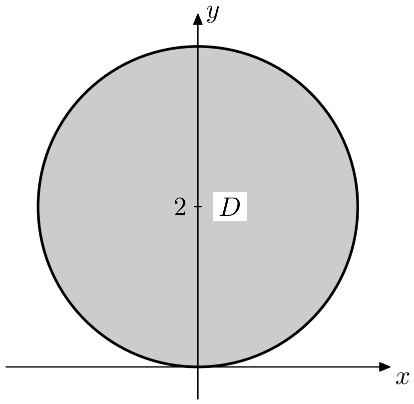

Figure 7.11. The disk $x^2 + y^2 \le 4y$

The x axis is the tangent line of this circle, hence, the polar angle changes from 0 to π, i.e. $0 \le \varphi \le \pi$. The least value of the polar radius is 0 for any polar angle. The greatest value of the polar radius depends on the polar angle. To get this dependence, we convert by (7.18) the equation of the circle $x^2 + y^2 = 4y$ into polar coordinates

$$\rho^2\cos^2\varphi + \rho^2\sin^2\varphi = 4\rho\sin\varphi$$

which yields

$$\rho = 4\sin\varphi$$

Thus in the disk given, the polar radius satisfies the inequalities $0 \le \rho \le 4\sin\varphi$.

By the formula (7.19) we get

$$\iint\limits_{D} f(x,y)dxdy = \iint\limits_{\Delta} f(\rho\cos\varphi, \rho\sin\varphi)\rho d\varphi d\rho$$

where Δ is determined by the inequalities $0 \le \varphi \le \pi$ and $0 \le \rho \le 4\sin\varphi$. Using the iterated integral, we obtain

$$\iint\limits_{D} f(x,y)dxdy = \int\limits_{0}^{\pi} d\varphi \int\limits_{0}^{4\sin\varphi} f(\rho\cos\varphi, \rho\sin\varphi)\rho d\rho$$

Example 2. Using the polar coordinates, compute the double integral

$$\iint\limits_{D} \frac{dxdy}{x^2+y^2+1}$$

if the region of integration D is bounded by $y=0$ and $y=\sqrt{1-x^2}$.

Since $x^2+y^2+1 = \rho^2\cos^2\varphi + \rho^2\sin^2\varphi + 1 = \rho^2+1$, then

$$\iint\limits_{D} \frac{dxdy}{x^2+y^2+1} = \iint\limits_{\Delta} \frac{\rho d\varphi d\rho}{\rho^2+1}$$

The region D is bounded by x axis and the upper half of the circle $x^2+y^2=1$.

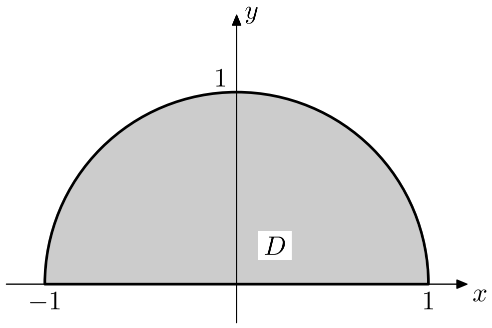

Figure 7.12. The half-disk bounded by x-axis and half-circle $y=\sqrt{1-x^2}$

The region D in Cartesian coordinates converts to the region Δ in polar coordinates, determined by the inequalities $0 \le \varphi \le \pi$ and $0 \le \rho \le 1$. Thus,

$$\iint\limits_{\Delta} \frac{\rho d\varphi d\rho}{\rho^2+1} = \int\limits_{0}^{\pi} d\varphi \int\limits_{0}^{1} \frac{\rho d\rho}{\rho^2+1}$$

First we find the inside integral. Since the differential of the denominator $d(\rho^2+1) = 2\rho d\rho$, then

$$\int_0^1 \frac{\rho d\rho}{\rho^2+1} = \frac{1}{2}\int_0^1 \frac{d(\rho^2+1)}{\rho^2+1} = \frac{1}{2}\ln(\rho^2+1)\Big|_0^1 = \frac{1}{2}\ln 2$$

Now, the outside integral

$$\int_0^{\pi} \frac{1}{2}\ln 2 d\varphi = \frac{1}{2}\ln 2\int_0^{\pi} d\varphi = \frac{\pi}{2}\ln 2$$

7.6 Computation of areas and volumes by double integrals

While defining the double integral we got the geometrical meaning of this. Assuming that $f(x,y) \geq 0$ in the region D, the double integral

$$\iint_D f(x,y)dxdy$$

is the volume of the solid enclosed by the region D in the xy plane, the graph of the function $z = f(x,y)$ and the cylinder surface, whose generatrix is parallel to z axis. This idea can be extended to more general regions.

Suppose that in the region D for two functions $f(x,y)$ and $g(x,y)$ there holds $f(x,y) \geq g(x,y)$. The property of the double integral gives

$$\iint_D [f(x,y) - g(x,y)]dxdy = \iint_D f(x,y)dxdy - \iint_D g(x,y)dxdy$$

Geometrically both integrals on the right mean the volumes of the solids. The first integral equals to the volume of the solid that lies below the surface $z = f(x,y)$ and above the region D in the xy plane. The second integral is the volume of the solid that lies between the surface $z = g(x,y)$ and the region D. Thus, the volume of the solid sketched in Figure 7.13 is computed by the formula

$$V = \iint_D [f(x,y) - g(x,y)]dxdy \tag{7.20}$$

This is the volume of the solid enclosed by the surface $z = f(x,y)$ from the top, by surface $z = g(x,y)$ from the bottom and the cylinder surface,

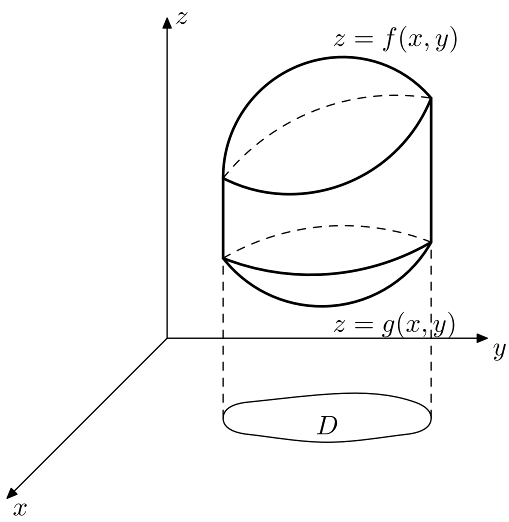

Figure 7.13.

whose directrix is the boundary of the region D and generatrix is parallel to z axis.

Example 1. Compute the volume of the solid enclosed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

The solid enclosed by these planes is the pyramid sketched in Figure 7.14. The pyramid is bounded by the plane $z = 1 - x - y$ from the top and by xy-plane $z = 0$ from the bottom. Using the formula (7.20) $f(x, y) = 1 - x - y$ and $g(x, y) = 0$. The volume of the pyramid is

$$V = \iint\limits_D (1 - x - y)dxdy$$

The region D is determined by inequalities $0 \le x \le 1$ and $0 \le y \le 1 - x$, thus,

$$V = \int\limits_0^1 dx \int\limits_0^{1-x} (1 - x - y)dy$$

First we compute the inside integral

$$\int\limits_0^{1-x} (1-x-y)dy = -\int\limits_0^{1-x} (1-x-y)d(1-x-y) = -\frac{(1-x-y)^2}{2}\Bigg|_0^{1-x} = \frac{(1-x)^2}{2}$$

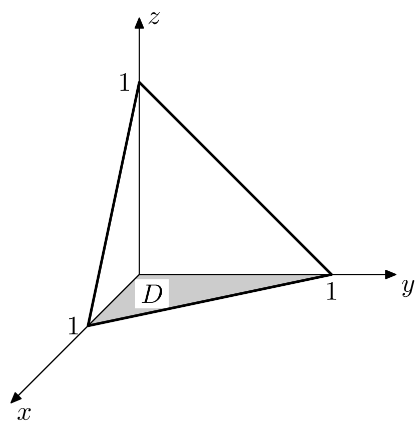

Figure 7.14. The solid in Example 1

and second the outside integral

$$V = \int_0^1 \frac{(1-x)^2}{2}dx = -\int_0^1 \frac{(1-x)^2}{2}d(1-x) = -\frac{(1-x)^3}{6}\Bigg|_0^1 = \frac{1}{6}$$

If the height of the solid $f(x,y) = 1$ at any point of the region D, then the volume of this solid $V = S_D \cdot 1$, where S_D is the area of the bottom (the region D). So, in this case the area of the bottom and the volume of the solid are numerically equal. Substituting the function $f(x,y) = 1$ into the formula

$$V = \iint_D f(x,y)dxdy$$

we get the formula to compute the area of the plane region D

$$S_D = \iint_D dxdy \tag{7.21}$$

Example 2. Compute the area of the region bounded by the lemniscate $(x^2+y^2)^2 = a^2(x^2-y^2)$.

Converting the equation of the lemniscate into polar coordinates, we obtain $\rho = a\sqrt{\cos 2\varphi}$. Hence, $-\frac{\pi}{2} \le 2\varphi \le \frac{\pi}{2}$, which yields $-\frac{\pi}{4} \le \varphi \le \frac{\pi}{4}$ or $\frac{3\pi}{4} \le \varphi \le \frac{5\pi}{4}$.

The lemniscate is symmetrical with respect to the x axis and y axis. Therefore we compute the area of the region D bounded by the lemniscate

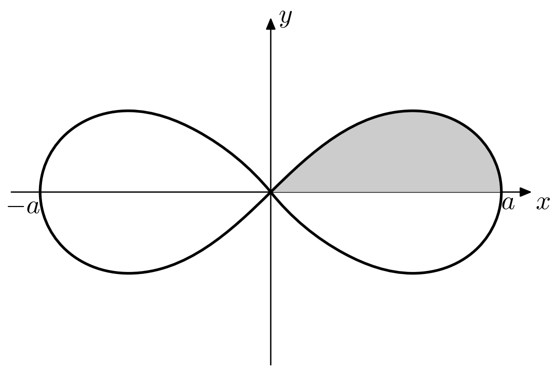

Figure 7.15. The lemniscate

in the first quadrant of the coordinate plane and multiply the result by 4. Converting the the formula (7.21) to polar coordinates gives

$$S = 4\iint\limits_{D} dxdy = 4\iint\limits_{\Delta} \rho d\varphi d\rho$$

The region of integration Δ is determined by the inequalities $0 \le \varphi \le \dfrac{\pi}{4}$ and $0 \le \rho \le a\sqrt{\cos 2\varphi}$. Hence,

$$S = 4\int\limits_{0}^{\frac{\pi}{4}} d\varphi \int\limits_{0}^{a\sqrt{\cos 2\varphi}} \rho d\rho$$

Computing the inside integral gives

$$\int\limits_{0}^{a\sqrt{\cos 2\varphi}} \rho d\rho = \frac{\rho^2}{2}\bigg|_{0}^{a\sqrt{\cos 2\varphi}} = \frac{a^2}{2}\cos 2\varphi$$

and the outside integral

$$S = 4\int\limits_{0}^{\frac{\pi}{4}} \frac{a^2}{2}\cos 2\varphi d\varphi = a^2\int\limits_{0}^{\frac{\pi}{4}} \cos 2\varphi d(2\varphi) = a^2 \sin 2\varphi\bigg|_{0}^{\frac{\pi}{4}} = a^2$$

7.7 Definition and properties of triple integral

We used a double integral to integrate over a two-dimensional region and so it's natural that we'll use a triple integral to integrate over a three dimensional region. Suppose the function of three variables $f(x, y, z)$ is defined in the three dimensional region V. Choose the whatever partition of the region V into n subregions

$$\Delta v_1,\ \Delta v_2,\ \ldots,\ \Delta v_k,\ \ldots, \Delta v_n$$

where Δv_k denotes the kth subregion, as well the volume of this subregion.

For each subregion, we pick a random point $P_k(\xi_k, \eta_k, \zeta_k) \in \Delta v_k$ to represent that subregion and find the product of the value of the function at that point and the volume of subregion $f(P_k)\Delta v_k$. Adding all these products, we obtain the sum

$$\sum_{k=0}^{n} f(P_k)\Delta v_k, \tag{7.22}$$

which is called the *integral sum* of the function $f(x, y, z)$ over the region V.

Let

$$\operatorname{diam} \Delta v_k = \max_{P,Q\in\Delta v_k} |\overrightarrow{PQ}|$$

be the diameter of the subregion Δv_k and λ the greatest diameter of the subregions, i.e.

$$\lambda = \max_{0\le k\le n} \operatorname{diam} \Delta v_k$$

Definition. If there exists the limit

$$\lim_{\lambda\to 0} \sum_{k=0}^{n} f(P_k)\Delta v_k$$

and this limit doesn't depend on the partition of the region V and the choice of the points P_k in the subregions, then this limit is called the triple integral of the function $f(x, y, z)$ over the region V and denoted

$$\iiint\limits_V f(x, y, z)dxdydz$$

Thus, by this definition

$$\iiint\limits_V f(x, y, z)dxdydz = \lim_{\lambda\to 0} \sum_{k=0}^{n} f(P_k)\Delta v_k \tag{7.23}$$

If the function $f(x, y, z)$ is continuous in the closed region V, then the triple integral (7.23) always exists.

The properties of the triple integral are quite similar to the properties of the double integral.

Property 1.

$$\iiint\limits_V [f(x,y,z) \pm g(x,y,z)]dxdydz = \iiint\limits_V f(x,y,z)dxdydz \pm \iiint\limits_V g(x,y,z)dxdydz$$

Property 2. If c is a constant then

$$\iiint\limits_V cf(x,y,z)dxdydz = c\iiint\limits_V f(x,y,z)dxdydz$$

Property 3. If $V = V_1 \cup V_2$ and the regions V_1 and V_2 have no common interior point then

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_{V_1} f(x,y,z)dxdydz + \iiint\limits_{V_2} f(x,y,z)dxdydz$$

Let $f(x,y,z) \geq 0$ be the density of a three-dimensional solid V at the point (x,y,z) inside the solid. By picking a point P_k to represent the subregion Δv_k we treat the density $f(P_k)$ constant in the subregion Δv_k and the product $f(P_k)\Delta v_k$ is the approximate mass of the subregion Δv_k. The approximate mass because we have substituted the variable density $f(x,y,z)$ by the constant density $f(P_k)$.

The integral sum is the sum of the approximate masses of the subregions, i.e. the approximate mass of the region V. The limiting process $\lambda \to 0$ means that all diameters of the subregions are infinitesimals. The density at the point P_k represents the density of the subregion Δv_k with the greater accuracy and the integral sum will approach to the total mass of the region V.

Therefore, if the function $f(x,y,z) \geq 0$ is the density of a three-dimensional solid V then the triple integral equals to the mass of the solid V

$$m = \iiint\limits_V f(x,y,z)dxdydz$$

If the region V has the uniform density 1, then the mass and volume are numerically equal, i.e. if $f(x,y,z) \equiv 1$, then the volume of the region V is computable by the formula

$$V = \iiint\limits_V dxdydz \tag{7.24}$$

An example how to use this formula we have later.

7.8 Evaluation of triple integral

The region V in the space is called *regular in direction of z axis* if there are satisfied three conditions.

1. Any line parallel to the z axis passing the interior point of this region cuts the boundary surface at two points.
2. The projection of the region onto xy plane is a regular plain region.
3. Cutting the region by the plane parallel to some coordinate plane creates two regions satisfying the conditions 1. and 2.

If those conditions are fulfilled, then the region V is determined by inequalities $a \leq x \leq b$, $\varphi_1(x) \leq y \leq \varphi_2(x)$ and $\psi_1(x,y) \leq z \leq \psi_2(x,y)$. We can define the iterated integral

$$I_V = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dy \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz$$

To compute this iterated integral we have to compute three definite integrals. First we integrate with respect to the variable z holding x and y constant

$$\Psi(x,y) = \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz$$

We call this inside integral and the result is a function of two variables $\Psi(x,y)$. Next we integrate with respect to the intermediate variable y holding x constant

$$\Phi(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} \Psi(x,y)dy$$

The result is a function of the one variable $\Phi(x)$. Finally we compute the outside integral

$$\int_a^b \Phi(x)dx$$

Notice that the limits of the outside variable a and b are always constants. The limits of the intermediate variable $\varphi_1(x)$ and $\varphi_2(x)$ depend in general on the outside variable. The limits of the inside variable $\psi_1(x,y)$ and $\psi_2(x,y)$

depend in general on the outside variable and on the intermediate variable.

Example 1. Compute the iterated integral $\int\limits_0^1 dx \int\limits_0^x dy \int\limits_0^{xy} (x+y)dz$

First we integrate with respect to the inner variable z. Since $x+y$ is constant, then

$$\int\limits_0^{xy} (x+y)dz = (x+y) \cdot z \Big|_0^{xy} = x^2y + xy^2$$

This result we integrate with respect to intermediate variable y

$$\int\limits_0^{x} (x^2y+xy^2)dy = x^2 \cdot \frac{y^2}{2}\Big|_0^x + x \cdot \frac{y^3}{}3\Big|_0^x = \frac{x^4}{2} + \frac{x^4}{3} = \frac{5x^4}{6}$$

and finally with respect to x

$$\int\limits_0^1 \frac{5x^4}{6} dx = \frac{5}{6} \cdot \frac{x^5}{5}\Big|_0^1 = \frac{1}{6}$$

Since we have assumed that the projection of the region V onto xy plane is a regular plane region, then the region V can be determined by inequalities $c \le y \le d$, $\varphi_1(y) \le x \le \varphi_2(y)$ and $\psi_1(x,y) \le z \le \psi_2(x,y)$ and the iterated integral can be defined as

$$I_V = \int\limits_c^d dy \int\limits_{\varphi_1(y)}^{\varphi_2(y)} dx \int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz$$

Just like we have defined the regular region in direction of z axis, we can define the regular region in direction of x axis and the regular region in direction of y axis. In the first case it is possible to define the iterated integrals

$$I_V = \int\limits_a^b dy \int\limits_{\varphi_1(y)}^{\varphi_2(y)} dz \int\limits_{\psi_1(y,z)}^{\psi_2(y,z)} f(x,y,z)dx$$

or

$$I_V = \int\limits_a^b dz \int\limits_{\varphi_1(z)}^{\varphi_2(z)} dy \int\limits_{\psi_1(x,y)}^{\psi_2(y,z)} f(x,y,z)dx$$

and in the second case

$$I_V = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dz \int_{\psi_1(x,z)}^{\psi_2(x,z)} f(x,y,z)dy$$

or

$$I_V = \int_a^b dz \int_{\varphi_1(z)}^{\varphi_2(z)} dx \int_{\psi_1(x,z)}^{\psi_2(x,z)} f(x,y,z)dy$$

So, if the region V is regular in direction of all coordinate axes, six orders of integration are possible. The conversion of the iterated integral for one order of integration to the iterated integral for another order of integration is called the change of the order of integration.

The iterated integral has the most simple limits, if the region of integration is a rectangular box defined by $a \le x \le b$, $c \le y \le d$ and $p \le z \le q$. All the faces of that box are parallel to one of three coordinate planes.

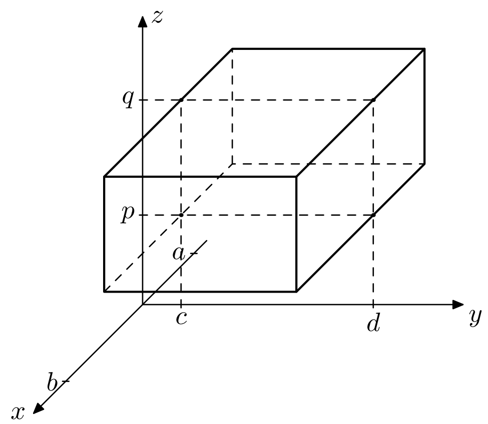

Figure 7.16. Rectangular box

If we choose x the outer variable, y the intermediate variable and z the inner variable, we compute

$$I_V = \int_a^b dx \int_c^d dy \int_p^q f(x,y,z)dy$$

and, of course, five more orders of integration are possible.

The iterated integral is the appropriate tool to compute the triple integral.
Theorem. If the function $f(x,y,z)$ is continuous in the closed regular region V, then

$$\iiint\limits_V f(x,y,z)dxdydz = \int\limits_a^b dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} dy \int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz \tag{7.25}$$

Example 2. Compute the triple integral

$$\iiint\limits_V xyzdxdydz$$

if the region V is bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

First three planes are the coordinate planes. The fourth plane passes three points $(1;0;0)$, $(0;1;0)$ and $(0;0;1)$. The intersection line of this plane and xy plane $z=0$ is $x+y=1$. The region of integration is sketched in Figure 7.17.

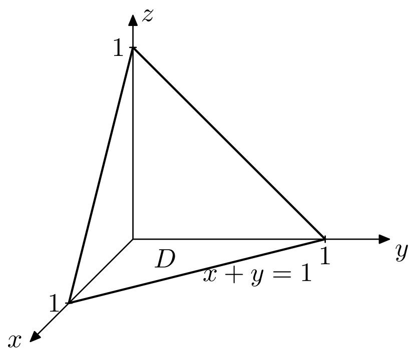

Figure 7.17. Region of integration of Example 2

The projection of the region of integration onto the xy plane is the triangle, which is determined by equalities $0 \le x \le 1$ and $0 \le y \le 1-x$. Since the region of integration is bounded by the plane $z=0$ on the bottom and by $z=1-x-y$ on the top, the region of integration is determined by $0 \le x \le 1$, $0 \le y \le 1-x$ and $0 \le z \le 1-x-y$. By the formula (7.25)

$$\iiint\limits_V xyzdxdydz = \int\limits_0^1 dx \int\limits_0^{1-x} dy \int\limits_0^{1-x-y} xyzdz$$

First we compute the inside integral

$$\int_0^{1-x-y} xyz dz = xy\frac{z^2}{2}\Big|_0^{1-x-y} = xy\frac{(1-x-y)^2}{2}$$

Next we integrate with respect to y

$$\int_0^{1-x} xy\frac{(1-x-y)^2}{2}dy = \frac{x}{2}\int_0^{1-x} y[(1-x)^2 - 2(1-x)y + y^2]dy$$

$$= \frac{x}{2}\left[(1-x)^2\frac{y^2}{2} - 2(1-x)\frac{y^3}{3} + \frac{y^4}{4}\right]\Big|_0^{1-x}$$

$$= \frac{x}{2}\left[\frac{(1-x)^4}{2} - \frac{2(1-x)^4}{3} + \frac{(1-x)^4}{4}\right] = \frac{x(1-x)^4}{24}$$

and finally

$$\frac{1}{24}\int_0^1 x(1-x)^4 dx = -\frac{1}{24}\int_0^1 (-x)(1-x)^4 dx$$

$$= -\frac{1}{24}\int_0^1 (1-x-1)(1-x)^4 dx = \frac{1}{24}\int_0^1 [(1-x)^5 - (1-x)^4]d(1-x)$$

$$= \frac{1}{24}\left[\frac{(1-x)^6}{6} - \frac{(1-x)^5}{5}\right]\Big|_0^1 = \frac{1}{24}(-\frac{1}{6}+\frac{1}{5}) = \frac{1}{720}$$

7.9 Change of variable in triple integral

Changing variables in triple integrals is nearly identical to changing variables in double integrals. We are going to change the variables in the triple integral

$$\iiint_V f(x,y,z)dxdydz$$

over the region V in the xyz space. We use the transformation

$$\begin{cases} x = \varphi(u,v,w) \\ y = \psi(u,v,w) \\ z = \chi(u,v,w) \end{cases} \tag{7.26}$$

to transform the region V into the new region V' in the uvw space. We assume that the functions x, y and z of the variables u, v and w are one-valued and the system of equations (7.26) has unique solution for u, v and w. Then to any point in the region V' there is related one point in the region V and vice versa. In addition we assume that the functions (7.26) are continuous and they have continuous partial derivatives with respect to all three variables in the region V'.

The jacobian of this change of variables is the determinant

$$J = \begin{vmatrix} x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \\ x'_w & y'_w & z'_w \end{vmatrix} \tag{7.27}$$

and we can transform the triple integral over the region V into the triple integral over the region V' by the formula

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_{V'} f(\varphi(u,v,w),\psi(u,v,w),\chi(u,v,w))|J|dudvdw \tag{7.28}$$

7.10 Triple integral in cylindrical coordinates

The cylindrical coordinates are really nothing more than an extension of polar coordinates into three dimensions leaving the z coordinate unchanged. For the given point $P(x,y,z)$ in the xyz space we denote P' the projection of this point onto xy plane. Denote by ρ the distance of P' from the origin and by φ the angle between the segment $P'O$ and x axis. Those φ and ρ are exactly the same as the polar coordinates in the two-dimensional case.

Definition. The cylindrical coordinates of the point P are called φ, ρ and z.

Since φ and ρ have in the xy plane the same meaning as the polar coordinates then the conversion formulas from the Cartesian coordinates into cylindrical coordinates are

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases} \tag{7.29}$$

Find the jacobian of this change of variables. By the formula (7.27) we get

$$J = \begin{vmatrix} x'_\varphi & y'_\varphi & z'_\varphi \\ x'_\rho & y'_\rho & z'_\rho \\ x'_z & y'_z & z'_z \end{vmatrix}$$

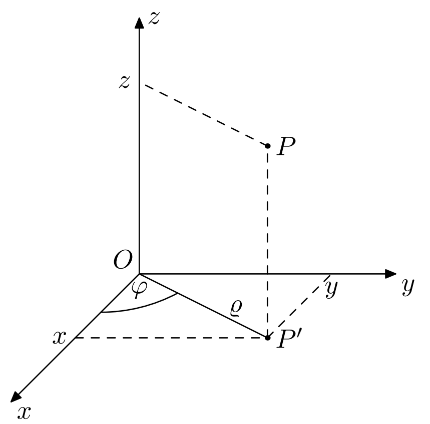

Figure 7.18. The cylindrical coordinates φ, ρ and z of the point P

The variable z does not depend on φ and ρ, hence, $z'_\varphi = 0$ and $z'_\rho = 0$. The variables x and y does not depend on z, i.e. $x'_z = 0$ and $y'_z = 0$. Consequently,

$$J = \begin{vmatrix} -\rho\sin\varphi & \rho\cos\varphi & 0 \\ \cos\varphi & \sin\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding this determinant by the last column gives

$$J = \begin{vmatrix} -\rho\sin\varphi & \rho\cos\varphi \\ \cos\varphi & \sin\varphi \end{vmatrix} = -\rho\sin^2\varphi - \rho\cos^2\varphi = -\rho.$$

Since ρ is a distance $|J| = \rho$.

Let V' be the region in cylindrical coordinates, which corresponds to the region V in Cartesian coordinates. By the general formula for change of variables in the triple integral (7.28) we obtain the formula to convert the triple integral in Cartesian coordinates into the triple integral in cylindrical coordinates

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_{V'} f(\rho\cos\varphi, \rho\sin\varphi, z)\rho d\varphi d\rho dz \tag{7.30}$$

Supposing that the region V' in cylindrical coordinates is given by the inequalities $\alpha \le \varphi \le \beta$, $\rho_1(\varphi) \le \rho \le \rho_2(\varphi)$ and $z_1(\varphi,\rho) \le z \le z_2(\varphi,\rho)$, we can write by the formula (7.25)

$$\iiint\limits_V f(x,y,z)dxdydz = \int\limits_\alpha^b d\varphi \int\limits_{\rho_1(\varphi)}^{\rho_2(\varphi)} d\rho \int\limits_{z_1(\varphi,\rho)}^{z_2(\varphi,\rho)} f(\rho\cos\varphi, \rho\sin\varphi, z)\rho dz \tag{7.31}$$

Example 1. Convert $\int\limits_{-1}^{1} dy \int\limits_{0}^{\sqrt{1-y^2}} dx \int\limits_{x^2+y^2}^{\sqrt{x^2+y^2}} f(x,y,z)dz$ into an integral in cylindrical coordinates.

The ranges of the variables in Cartesian coordinates from this iterated integral are

$$-1 \leq y \leq 1$$
$$0 \leq x \leq \sqrt{1-y^2}$$
$$x^2 + y^2 \leq z \leq \sqrt{x^2+y^2}$$

The first two inequalities define the projection D of this region onto xy-plane, which is the half of the disk of radius 1 centered at the origin. The third equality determines that the region of integration is bounded by the paraboloid of rotation $z = x^2 + y^2$ on the bottom and by the cone $z = \sqrt{x^2+y^2}$ on the top. The region of integration and its projection onto xy plane are in Figure 7.19.

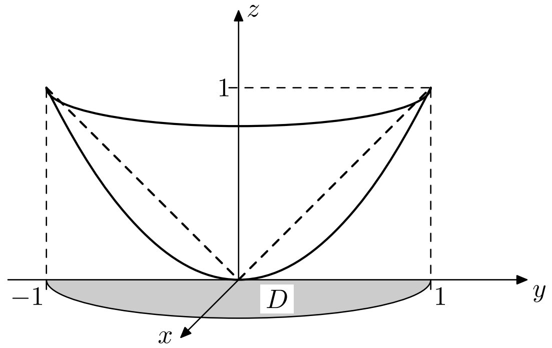

Figure 7.19. Region of integration of Example 1

In cylindrical coordinates the equation on the paraboloid of rotation converts to $z = \rho^2$ and the equation of the cone to $z = \rho$. So, the ranges for the region of integration in cylindrical coordinates are,

$$-\tfrac{\pi}{2} \leq \varphi \leq \tfrac{\pi}{2}$$
$$0 \leq \rho \leq 1$$
$$\rho^2 \leq z \leq \rho$$

Now, by the formula (7.31) we write

$$\int\limits_{-1}^{1} dy \int\limits_{0}^{\sqrt{1-y^2}} dx \int\limits_{x^2+y^2}^{\sqrt{x^2+y^2}} f(x,y,z)dz = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_{0}^{1} d\rho \int\limits_{\rho^2}^{\rho} f(\rho\cos\rho, \rho\sin\varphi, z)\rho dz$$

Notice that the limits of integration are simpler in the cylindrical coordinates.

Example 2. Using the cylindrical coordinate, compute the triple integral

$$\int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2+y^2}dz$$

In Cartesian coordinates the region of integration is defined by the inequalities $0 \le x \le 2$, $0 \le y \le \sqrt{2x-x^2}$ and $0 \le z \le a$, i.e. bounded by the planes $x = 0$, $x = 2$, $y = 0$, $z = 0$ and $z = a$ and by the cylinder $y = \sqrt{2x-x^2}$. The generatrix of the cylinder is parallel to the z axis and the projection onto xy plane is the half circle $y = \sqrt{2x-x^2}$. This is the upper half of the circle $y^2 = 2x - x^2$ or $x^2 - 2x + y^2 = 0$, i.e.

$$(x-1)^2 + y^2 = 1$$

which is the circle of radius 1 centered at $(1;0)$. The region of integration is sketched in Figure 7.20.

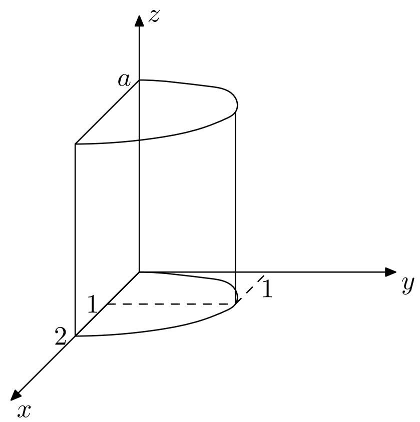

Figure 7.20. The region of integration of Example 2

Convert the integral given into the integral in cylindrical coordinates. The range of the angle φ in the projection of this region onto xy plane is $0 \le \varphi \le \dfrac{\pi}{2}$. Converting the equation of the cylinder $x^2 + y^2 = 2x$ into cylindrical coordinates gives $\rho^2 \cos^2\varphi + \rho^2 \sin^2\varphi = 2\rho\cos\varphi$ or $\rho = 2\cos\varphi$. Hence, the range for ρ is $0 \le \rho \le 2\cos\varphi$. We didn't convert the third coordinate z, thus, $0 \le z \le a$.

Converting the integrand into cylindrical coordinates gives

$$z\sqrt{\rho^2\cos^2\varphi + \rho^2\sin^2\varphi} = z\rho$$

Now, by the formula by (7.31)

$$\int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2+y^2}dz = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} d\rho \int_0^a z\rho \cdot \rho dz$$

The integration with respect to z gives

$$\int_0^a z\rho^2 dz = \rho^2 \frac{z^2}{2}\bigg|_0^a = \frac{a^2\rho^2}{2}$$

the integration with respect to ρ gives

$$\frac{a^2}{2}\int_0^{2\cos\varphi} \rho^2 d\varrho = \frac{a^2}{2}\frac{\rho^3}{3}\bigg|_0^{2\cos\varphi} = \frac{4a^2\cos^3\varphi}{3}$$

Finally, integrating with respect to φ, we get

$$\frac{4a^2}{3}\int_0^{\frac{\pi}{2}} \cos^3\varphi d\varphi = \frac{4a^2}{3}\int_0^{\frac{\pi}{2}} (1-\sin^2\varphi)d(\sin\varphi) = \frac{4a^2}{3}\left(\sin\varphi - \frac{\sin^3\varphi}{3}\right)\bigg|_0^{\frac{\pi}{2}} = \frac{8a^2}{9}.$$

7.11 Triple integral in spherical coordinates

In the previous subsection we looked at computing integrals in terms of cylindrical coordinates and we now take a look at computing integrals in terms of spherical coordinates.

First, we have to define the spherical coordinates. The following sketch shows the relationship between the Cartesian and spherical coordinate systems.

For a point $P(x, y, z)$ in Cartesian coordinates given denote by P' the projection of this point onto xy plane. Denote by φ the angle between the segment OP' and x axis, by θ the angle between OP and z axis and by r the distance of the point P from the origin, i.e. the length of segment OP. Any point in the space is uniquely determine by three parameters φ, θ and r. In Figure 7.21 the point P has been chosen in the first octant, hence all three coordinates are positive. The abscissa x of the point P is the length of segment OR, the ordinate y the length of segment $RP'$and third coordinate z the length of segment OQ.

Using the right triangle OQP, we get

$$\sin\theta = \frac{QP}{r}$$

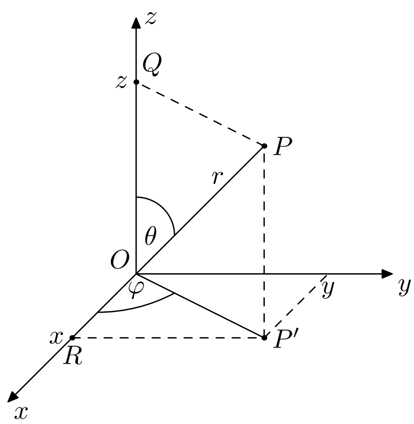

Figure 7.21. The spherical coordinates φ, θ and r of the point P

and

$$\cos\theta = \frac{z}{r}$$

Since $OP' = QP$, then the first equality gives $OP' = r\sin\theta$. The second equality gives $z = r\cos\theta$.

From the right triangle ORP' we get $\cos\varphi = \dfrac{x}{OP'}$ and $\sin\varphi = \dfrac{y}{OP'}$ which yield

$$x = OP'\cos\varphi$$

and

$$y = OP'\sin\varphi$$

Substituting OP' into the formulas for x and y, we obtain the conversion formulas from Cartesian coordinates into spherical coordinates

$$\begin{cases} x = r\cos\varphi\sin\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\theta \end{cases} \tag{7.32}$$

Find the jacobian for this change of variable. By the (7.27) we obtain

$$J = \begin{vmatrix} x'_\varphi & y'_\varphi & z'_\varphi \\ x'_\theta & y'_\theta & z'_\theta \\ x'_r & y'_r & z'_r \end{vmatrix} = \begin{vmatrix} -r\sin\varphi\sin\theta & r\cos\varphi\sin\theta & 0 \\ r\cos\varphi\cos\theta & r\sin\varphi\cos\theta & -r\sin\theta \\ \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \end{vmatrix}$$

Expanding this determinant, we get

$$J = -r^2\sin^2\varphi\sin\theta\cos^2\theta - r^2\cos^2\varphi\sin^3\theta - r^2\sin^2\varphi\sin^3\theta - r^2\cos^2\varphi\sin\theta\cos^2\theta$$

Adding the 1st and 4th and the 2nd and 3rd term gives

$$\begin{aligned} J &= -r^2\sin\theta\cos^2\theta(\sin^2\varphi+\cos^2\varphi)-r^2\sin^3\theta(\cos^2\varphi+\sin^2\varphi) \\ &= -r^2\sin\theta(\cos^2\theta+\sin^2\theta)=-r^2\sin\theta \end{aligned}$$

In spherical coordinates the angle θ is measured with respect to z axis. Hence, we have the restriction $0\le\theta\le\pi$. Then $\sin\theta\ge 0$ and the absolute value of jacobian is

$$|J|=r^2\sin\theta$$

Let V' denotes the region in spherical coordinates corresponding to the region V in Cartesian coordinates. By the formula (7.28) we get the formula to convert the triple integral into the spherical coordinates

$$\iiint\limits_V f(x,y,z)dxdydz=\iiint\limits_{V'} f(r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)r^2\sin\theta d\varphi d\theta dr. \tag{7.33}$$

The ranges for the variables φ, θ and r are

$$\begin{cases} \alpha\le\varphi\le\beta \\ \gamma\le\varphi\le\delta \\ r_1(\varphi,\theta)\le r\le r_2(\varphi,\theta) \end{cases}$$

Therefore the integral (7.33) will become,

$$\iiint\limits_V f(x,y,z)dxdydz=\int\limits_\alpha^\beta d\varphi\int\limits_\gamma^\delta d\theta\int\limits_{r_1(\varphi,\theta)}^{r_2(\varphi,\theta)} f(r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)r^2\sin\theta dr. \tag{7.34}$$

First of all we use the spherical coordinates to compute the triple integral, if the region of integration is ball or some portion of the ball.

Example 1. Compute the volume of the intersection of balls bounded by the spheres $x^2+y^2+z^2=R^2$ and $x^2+y^2+z^2=2Rz$.

We are going to use the formula

$$V=\iiint\limits_V dxdydz$$

The first sphere is the sphere of radius R centered at the origin. Converting the equation of the second sphere to $x^2+y^2+(z-R)^2=R^2$, we see that this is the sphere of radius R centered at $B(0;0;R)$.

The intersection of the two balls is the solid of rotation, with z axis as the axis of rotation. Thus, the angle φ makes the full rotation, i.e. $0\le\varphi\le 2\pi$.

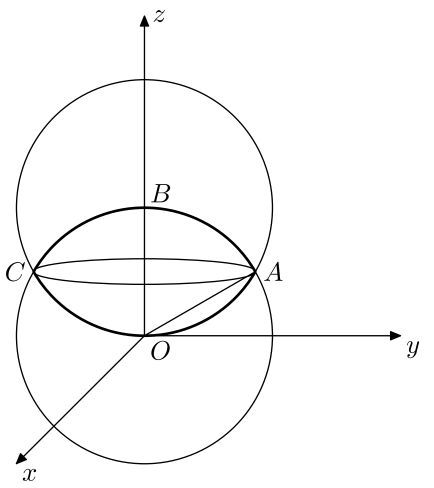

Figure 7.22. The intersection of the balls

Converting the sum of the squares $x^2+y^2+z^2$ into spherical coordinates, we obtain

$$\begin{aligned} r^2\cos^2\varphi\sin^2\theta + r^2\sin^2\varphi\sin^2\theta + r^2\cos^2\theta &= \\ = r^2\sin^2\theta(\cos^2\varphi+\sin^2\varphi) + r^2\cos^2\theta &= \\ = r^2\sin^2\theta + r^2\cos^2\theta = r^2(\sin^2\theta+\cos^2\theta) &= r^2 \end{aligned}$$

Consequently, the equation of the sphere $x^2+y^2+z^2=R^2$ is in the spherical coordinates

$$r = R$$

and the equation of the sphere $x^2+y^2+z^2=2Rz$

$$r = 2R\cos\theta$$

Since the maximal distance is different on these spheres, we have to divide the region of integration V into two subregions $V = V_1 \cup_2 V$. In the first subregion θ rotates from the position OB to the position OA, hence, $0 \le \theta \le \dfrac{\pi}{3}$. In the second subregion θ rotates from the position OA to the y axis, hence, $\dfrac{\pi}{3} \le \theta \le \dfrac{\pi}{2}$.

In the first subregion V_1 the limits for the variables are,

$$0 \le \varphi \le 2\pi$$

$$0 \le \theta \le \frac{\pi}{3}$$

$$0 \le r \le R$$

and in the second subregion V_2,

$$0 \le \varphi \le 2\pi$$

$$\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$$

$$0 \le r \le 2R\cos\theta$$

The volume of the intersection of two balls is

$$\begin{aligned} V &= \iiint\limits_V dxdydz = \iiint\limits_{V_1} dxdydz + \iiint\limits_{V_2} dxdydz = \\ &= \int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{\pi}{3}} d\theta \int\limits_0^R r^2 \sin\theta dr + \int\limits_0^{2\pi} d\varphi \int\limits_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta \int\limits_0^{2R\cos\theta} r^2 \sin\theta dr \end{aligned}$$

We start the computation of the first addend. The integration with respect to r gives

$$\int\limits_0^R r^2 \sin\theta dr = \sin\theta \cdot \left.\frac{r^3}{3}\right|_0^R = \frac{R^3}{3}\sin\theta$$

the integration with respect to θ,

$$\int\limits_0^{\frac{\pi}{3}} \frac{R^3}{3}\sin\theta = -\frac{R^3}{3}\cos\theta\Big|_0^{\frac{\pi}{3}} = -\frac{R^3}{3}\left(\frac{1}{2} - 1\right) = \frac{R^3}{6}$$

and the integration with respect to φ,

$$\int\limits_0^{2\pi} \frac{R^3}{6} d\varphi = \frac{R^3}{6} \cdot \varphi\Big|_0^{2\pi} = \frac{\pi R^3}{3}$$

Integrating the second addend with resect to r, we get

$$\int\limits_0^{2R\cos\theta} r^2 \sin\theta dr = \sin\theta \cdot \left.\frac{r^3}{3}\right|_0^{2R\cos\theta} = \frac{8R^3}{3}\sin\theta\cos^3\theta$$

integrating with respect to θ, we get

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{8R^3}{3} \sin\theta \cos^3\theta d\theta = -\frac{8R^3}{3} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^3\theta d(\cos\theta)$$

$$= \quad -\frac{8R^3}{3}\frac{\cos^4\theta}{4}\bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = -\frac{8R^3}{3}\left(0 - \frac{1}{16\cdot 4}\right) = \frac{R^3}{24}$$

an integrating with respect to φ gives

$$\int_0^{2\pi} \frac{R^3}{24} d\varphi = \frac{R^3}{24}\cdot\varphi\bigg|_0^{2\pi} = \frac{\pi R^3}{12}$$

If we add these two results, we obtain the volume of this solid of rotation

$$V = \frac{\pi R^3}{3} + \frac{\pi R^3}{12} = \frac{5\pi R^3}{12}$$