8 Line and surface integrals

Line integral is an integral where the function to be integrated is evaluated along a curve. The terms path integral, curve integral, and curvilinear integral are also used.

8.1 Line integral with respect to arc length

Suppose that on the plane curve AB there is defined a function of two variables $f(x, y)$, i.e. to any point (x, y) of this curve there is related the value $f(x, y)$. Let

$$A = P_0,\ P_1,\ P_2,\ \ldots, P_{k-1},\ P_k,\ \ldots,\ P_n = B$$

the random partition of the curve AB into subarcs $\widehat{P_{k-1}P_k}$, $k = 1,\ 2,\ \ldots,\ n$. From every subarc we pick a random point $Q_k(\xi_k, \eta_k) \in \widehat{P_{k-1}P_k}$.

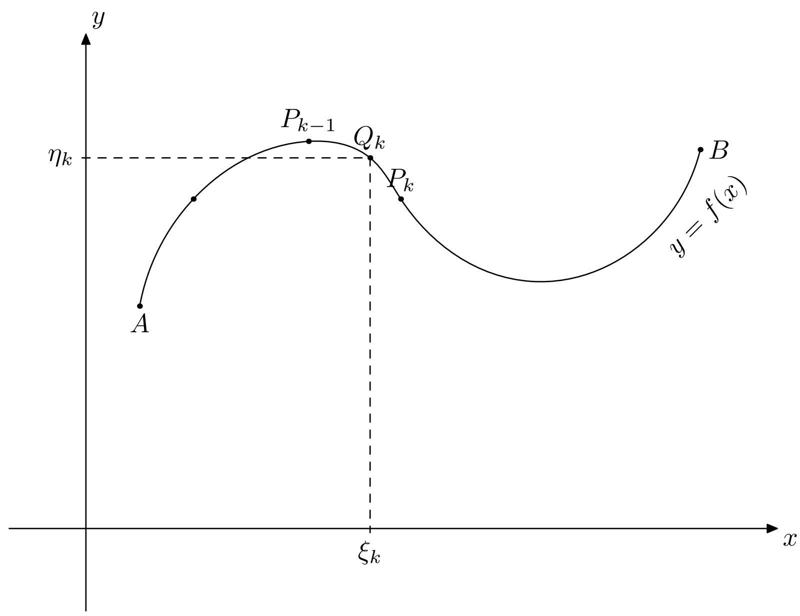

Figure 8.1. The partition of the curve AB

Denote by Δs_k the length of the subarc $\widehat{P_{k-1}P_k}$. Now we multiply the value at the point chosen by the length of subarc $f(Q_k)\Delta s_k$, where $k = 1,\ 2,\ \ldots,\ n$. Adding all those products, we get the sum

$$s_n = \sum_{k=1}^{n} f(Q_k)\Delta s_k \tag{8.1}$$

which is called the *integral sum* of the function $f(x, y)$ *over the curve* AB.

We have the random partition of the curve AB. Therefore, the lengths Δs_k of subarcs $\widehat{P_{k-1}P_k}$ are different. Denote by

$$\lambda = \max_{1 \leq k \leq n} \Delta s_k$$

i.e. the greatest length of subarcs.

Definition. If there exists the limit

$$\lim_{\lambda \to 0} s_n$$

and this limit does not depend on the partition of AB and does not depend on the choice of the points Q_k on the subarcs, then this limit is called the *line integral with respect to arc length* and denoted by

$$\int_{AB} f(x, y) ds$$

Thus, by the definition

$$\int_{AB} f(x, y) ds = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(Q_k) \Delta s_k$$

Suppose the curve AB is the piece of wire. If the function $\rho(x, y) \geq 0$ represents the density (mass per unit length) for wire AB, then the product $\rho(Q_k)\Delta s_k$ is the approximate mass of subarc Δs_k and the integral sum

$$\sum_{k=1}^{n} \rho(Q_k) \Delta s_k$$

is the approximate mass of the wire AB. For shorter subarc the value $\rho(Q_k)$ represents the variable density $\rho(x, y)$ of subarc with greater accuracy. Thus, in this case the limit of the integral sum, i.e. the line integral with respect to arc length gives the mass of the wire AB:

$$m = \int_{AB} \rho(x, y) ds \tag{8.2}$$

The properties on the line integral with respect to arc length can be proved directly, using the definition.

Property 1. The line integral with respect to arc length does not depend on the direction the curve AB has been traversed:

$$\int\limits_{AB} f(x,y)ds = \int\limits_{BA} f(x,y)ds$$

Property 2. (Additivity property) If C is some point on the curve AB, then

$$\int\limits_{AB} f(x,y)ds = \int\limits_{AC} f(x,y)ds + \int\limits_{CB} f(x,y)ds$$

Property 3.

$$\int\limits_{AB} [f(x,y) \pm g(x,y)]ds = \int\limits_{AB} f(x,y)ds \pm \int\limits_{AB} g(x,y)ds$$

Property 4. If c ic a constant, then

$$\int\limits_{AB} cf(x,y)ds = c\int\limits_{AB} f(x,y)ds$$

Property 5. Taking in the definition of the line integral with respect to arc length $f(x,y) \equiv 1$, we get the integral sum

$$s_n = \sum_{k=1}^{n} \Delta s_k$$

which is the sum of lengths of subarcs. This is the length of arc AB for any partition. Thus, for $f(x,y) \equiv 1$ the line integral gives us the length of arc AB:

$$s_{AB} = \int\limits_{AB} ds$$

Property 5 can be also obtained by taking in (8.2) the density $\rho(x,y) \equiv 1$ because then the mass and the length of the curve are numerically equal.

Any point of the curve AB in the space has three coordinates $Q_k(\xi_k, \eta_k, \zeta_k)$. So, the function defined on the space curve is a function of three variables $f(x,y,z)$. Defining the line integral with respect to arc length along the space curve we do everything like we did in the definition for the two-dimensional case:

$$\int\limits_{AB} f(x,y,z)ds = \lim_{\lambda\to 0} \sum_{k=1}^{n} f(Q_k)\Delta s_k \qquad (8.3)$$

Of course, five properties of the line integral for three-dimensional case are still valid.

8.2 Evaluation of line integral with respect to arc length

Suppose that the parametric equations of the curve AB in the plain are

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

and the parametric equations of the curve AB in the space are

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t), \end{cases}$$

where at the point A the value of the parameter $t = \alpha$ and at the point B the value of the parameter $t = \beta$.

Definition 1. The plain curve AB is called smooth, if $\dot{x} = \dfrac{dx}{dt}$ and $\dot{y} = \dfrac{dy}{dt}$ are continuous on $[\alpha; \beta]$ and

$$\dot{x}^2 + \dot{y}^2 \neq 0$$

Definition 2. The curve AB in the space is called smooth, if $\dot{x} = \dfrac{dx}{dt}$, $\dot{y} = \dfrac{dy}{dt}$ and $\dot{z} = \dfrac{dz}{dt}$ are continuous on $[\alpha; \beta]$ and

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \neq 0$$

Intuitively, a smooth curve is one that does not have sharp corners. The next theorems remain without proof.

Theorem 1. If the function $f(x, y)$ is continuous on the smooth curve AB, then

$$\int\limits_{AB} f(x, y)ds = \int\limits_{\alpha}^{\beta} f[x(t), y(t)]\sqrt{\dot{x}^2 + \dot{y}^2}dt \tag{8.4}$$

Theorem 2. If the function $f(x, y, z)$ is continuous on the smooth curve AB, then

$$\int\limits_{AB} f(x, y, z)ds = \int\limits_{\alpha}^{\beta} f[x(t), y(t), z(t)]\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}dt \tag{8.5}$$

If $\mathbf{r}(t) = (x(t), y(t), z(t))$ is the position vector of a point on the curve, then the square root in the formula (8.5) is the length of $\dot{\mathbf{r}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$ i.e $|\dot{\mathbf{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. The formula (8.5) can be re-written as

$$\int\limits_{AB} f(x, y, z)ds = \int\limits_{\alpha}^{\beta} f[x(t), y(t), z(t)]|\dot{\mathbf{r}}(t)|dt$$

Suppose the curve AB is a graph of the function $y = \varphi(x)$ given explicitly, at the point A $x = a$ and at B $x = b$. The curve is smooth, if there exists $\varphi'(x)$ on the interval $[a; b]$.

Theorem 3. If the function $f(x, y)$ is continuous on the smooth curve AB, then

$$\int\limits_{AB} f(x, y)ds = \int\limits_{a}^{b} f[x, \varphi(x)]\sqrt{1 + y'^2}dx \qquad (8.6)$$

This theorem is the direct conclusion of Theorem 1 because treating the variable x as the parameter, we have $\dot{x} = 1$ and $\dot{y} = \dfrac{dy}{dx} = y'$.

Example 1. Compute the line integral $\displaystyle\int\limits_{AB} \frac{ds}{x - y}$, where AB is the segment of the line $y = 2x - 3$ between coordinate axes.

The line is the graph of the function given explicitly. Therefore, we use for the computation the formula (8.6).

At the intersection point by y axis $x = 0$ and at the intersection point by x axis $y = 0$, i.e. $x = \dfrac{3}{2}$. To apply the formula, we find $y = 2$ and $1 + y'^2 = 5$. Thus,

$$\begin{aligned} \int\limits_{AB} \frac{ds}{x - y} &= \int\limits_{0}^{\frac{3}{2}} \frac{\sqrt{5}dx}{x - (2x - 3)} = \sqrt{5}\int\limits_{0}^{\frac{3}{2}} \frac{dx}{3 - x} = -\sqrt{5}\int\limits_{0}^{\frac{3}{2}} \frac{d(3 - x)}{3 - x} \\ &= -\sqrt{5}\ln|3 - x|\Big|_{0}^{\frac{3}{2}} = -\sqrt{5}\left(\ln\frac{3}{2} - \ln 3\right) = -\sqrt{5}\ln\frac{1}{2} = \sqrt{5}\ln 2 \end{aligned}$$

Example 2. Compute the line integral $\displaystyle\int\limits_{AB} \sqrt{y}ds$, where AB is the first arc of cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

For the first arc of cycloid $0 \le t \le 2\pi$. To apply the formula (8.4), we find $\dot{x} = a(1 - \cos t)$, $\dot{y} = a\sin t$ and

$$\dot{x}^2 + \dot{y}^2 = a^2(1 - \cos t)^2 + a^2\sin^2 t = a^2(1 - 2\cos t + \cos^2 t + \sin^2 t) = 2a^2(1 - \cos t)$$

By the formula (8.4)

$$\int\limits_{AB} \sqrt{y}ds = \int\limits_0^{2\pi} \sqrt{a(1-\cos t)}\sqrt{2a^2(1-\cos t)}dt =$$

$$a\sqrt{2a}\int\limits_0^{2\pi}(1-\cos t)dt = a\sqrt{2a}(t-\sin t)\Big|_0^{2\pi} = 2\pi a\sqrt{2a}$$

Example 3. Compute the line integral $\int\limits_{AB}(2z-\sqrt{x^2+y^2})ds$, where AB is the first turn of conical helix $x = t\cos t$, $y = t\sin t$, $z = t$.

For the first turn of conical helix $0 \le t \le 2\pi$. Find $\dot{x} = \cos t - t\sin t$, $\dot{y} = \sin t + t\cos t$, $\dot{z} = 1$ and

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1 =$$

$$\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 1 = 2 + t^2$$

By the formula (8.5) we obtain

$$\int\limits_{AB}(2z-\sqrt{x^2+y^2})ds = \int\limits_0^{2\pi}(2t-\sqrt{t^2\cos^2 t + t^2\sin^2 t})\sqrt{2+t^2}dt =$$

$$\int\limits_0^{2\pi}(2t-t)\sqrt{2+t^2}dt = \int\limits_0^{2\pi} t\sqrt{2+t^2}dt = \frac{1}{2}\int\limits_0^{2\pi}\sqrt{2+t^2}d(2+t^2) =$$

$$\frac{1}{2}\frac{(2+t^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{2\pi} = \frac{(2+t^2)^{\frac{3}{2}}}{3}\Bigg|_0^{2\pi} = \frac{(2+4\pi^2)\sqrt{2+4\pi^2}-2\sqrt{2}}{3}$$

8.3 Line integral with respect to coordinates

In the first subsection we defined the line integral for the scalar field. Now we are going to define the line integral for the vector field. First we consider the two-dimensional case. Let AB be the curve in the plain and $\overrightarrow{F} = (X(x,y); Y(x,y))$ a force vector. Suppose that the force is applied to an object to move it along the curve AB. The goal in to find the work done by this force. To do it, we first divide the curve AB with the points

$$A = P_0,\ P_1,\ \ldots,\ P_{k-1},\ P_k,\ \ldots,\ P_n = B$$

into subarcs $\widehat{P_{k-1}P_k}$, where $k = 1,\ 2,\ \ldots,\ n$ and approximate any subarc $\widehat{P_{k-1}P_k}$ to the vector $\overrightarrow{P_{k-1}P_k}$.

Denote the coordinates of the kth partition point P_k by x_k and y_k, i.e. $P_k(x_k; y_k)$ and the coordinates of the vector $\overrightarrow{P_{k-1}P_k}$ by

$$\Delta x_k = x_k - x_{k-1}$$

and

$$\Delta y_x = y_k - y_{k-1}$$

that is

$$\overrightarrow{P_{k-1}P_k} = (\Delta x_k; \Delta y_k)$$

Let Δs_k be the magnitude of the vector $\overrightarrow{P_{k-1}P_k}$:

$$\Delta s_k = \sqrt{\Delta x_k^2 + \Delta y_k^2}$$

and λ the greatest of all those magnitudes

$$\lambda = \max_{1 \le k \le n} \Delta s_k$$

Next we choose a random point $Q_k(\xi_k; \eta_k)$ on any subarc $\widehat{P_{k-1}P_k}$ and substitute on this subarc the variable force vector by the constant force vector

$$\overrightarrow{F_k} = (X(\xi_k, \eta_k); Y(\xi_k, \eta_k))$$

Recall that if a constant force $\overrightarrow{F_k}$ is applied to an object to move it along a straight line from the point P_{k-1} to the point P_k, then the amount of work done A_k is the scalar product of the force vector and the vector $\overrightarrow{P_{k-1}P_k}$:

$$A_k = \overrightarrow{F_k} \cdot \overrightarrow{P_{k-1}P_k} = X(\xi_k, \eta_k)\Delta x_k + Y(\xi_k, \eta_k)\Delta y_k$$

The total work done by the force vector $\overrightarrow{F}$, moving an object from the point A to the point B along the curve is approximately

$$\sum_{k=1}^{n} [X(\xi_k, \eta_k)\Delta x_k + Y(\xi_k, \eta_k)\Delta y_k]. \tag{8.7}$$

Approximately because we have approximated the subarc $\widehat{P_{k-1}P_k}$ to the vector $\overrightarrow{P_{k-1}P_k}$ and the variable force vector $\overrightarrow{F} = (X(x,y); Y(x,y))$ to the constant vector $\overrightarrow{F_k} = (X(\xi_k, \eta_k); Y(\xi_k, \eta_k))$.

Obviously, taking more partition points, the subarcs get shorter and the vectors $\overrightarrow{P_{k-1}P_k}$ represent the subarcs $\widehat{P_{k-1}P_k}$ with greater accuracy. As well, the constant vector $\overrightarrow{F_k} = (X(\xi_k, \eta_k); Y(\xi_k, \eta_k))$ represents the variable vector $\overrightarrow{F} = (X(x, y); Y(x, y))$ on $\widehat{P_{k-1}P_k}$ with greater accuracy.

Definition. If the sum (8.7) has the limit as $\max \Delta s_k \to 0$ and this limit does not depend on the partition of the curve AB and does not depend on the choice of points Q_k on subarcs, then this limit is called the *line integral with respect to coordinates* and denoted

$$\int\limits_{AB} X(x, y)dx + Y(x, y)dy$$

Thus, by the definition

$$\int\limits_{AB} X(x, y)dx + Y(x, y)dy = \lim_{\lambda \to 0} \sum_{k=1}^{n} [X(\xi_k, \eta_k)\Delta x_k + Y(\xi_k, \eta_k)\Delta y_k] \qquad (8.8)$$

If AB is a curve in the space, then

$$\overrightarrow{P_{k-1}P_k} = (\Delta x_k; \Delta y_k; \Delta z_k)$$

and the magnitude of this vector

$$\Delta s_k = \sqrt{\Delta x_k^2 + \Delta y_k^2 + \Delta z_k^2}$$

Also the force vector has three coordinates

$$\overrightarrow{F} = (X(x, y, z); Y(x, y, z)); Z(x, y, z))$$

The line integral with respect to coordinates is defined as the limit

$$\int\limits_{AB} X(x, y, z)dx + Y(x, y, z)dy + Z(x, y, z)dz$$

$$= \lim_{\lambda \to 0} \sum_{k=1}^{n} [X(\xi_k, \eta_k, \zeta_k)\Delta x_k + Y(\xi_k, \eta_k, \zeta_k)\Delta y_k + Z(\xi_k, \eta_k, \zeta_k)\Delta z_k]$$

We consider the properties of the line integral with respect to coordinates for the curve in the plane. All of this discussion generalizes to space curves in a straightforward manner.

Property 1. If C is a random point on the curve AB, then

$$\int\limits_{AB} X(x, y)dx + Y(x, y)dy = \int\limits_{AC} X(x, y)dx + Y(x, y)dy + \int\limits_{CB} X(x, y)dx + Y(x, y)dy \qquad (8.9)$$

To prove this property it is enough when starting to define the line integral we choose C as the first partition point. The further random partition of the curve AB creates random partitions into the subarcs for the curves AC and CB. The integral sum over the curve AB is equal to the sum of integral sums over the curves AC and CB

$$\begin{aligned}\sum_{AB}[X(Q_k)\Delta x_k + Y(Q_k)\Delta y_k] &= \sum_{AC}[X(Q_k)\Delta x_k + Y(Q_k)\Delta y_k] + \\ &+ \sum_{CD}[X(Q_k)\Delta x_k + Y(Q_k)\Delta y_k]\end{aligned}$$

Finally, the limit of the sum is the sum of the limits.

Property 2. If the curve is traced in reverse (that is, from the terminal point to the initial point), then the sign of the line integral is reversed as well:

$$\int\limits_{BA} X(x,y)dx + Y(x,y)dy = -\int\limits_{AB} X(x,y)dx + Y(x,y)dy \qquad (8.10)$$

Proof. If we define the line integral traversing the curve in direction BA, we choose the same partition points, which we have chosen in the definition of the line integral in direction AB. Then instead of the vectors $\overrightarrow{P_{k-1}P_k}$ we have opposite vectors $\overrightarrow{P_kP_{k-1}} = (-\Delta x_k; -\Delta y_k)$ and at the point $Q_k(\xi_k;\eta_k)$ picked on the kth subarc the force vector is

$$\overrightarrow{F_k} = (X(\xi_k,\eta_k); Y(\xi_k,\eta_k))$$

Finding the limit as $\lambda \to 0$ of the integral sum

$$\sum_{k=1}^{n}[X(\xi_k,\eta_k)(-\Delta x_k)+Y(\xi_k,\eta_k)(-\Delta y_k)] = -\sum_{k=1}^{n}[X(\xi_k,\eta_k)\Delta x_k+Y(\xi_k,\eta_k)\Delta y_k]$$

completes the proof.

8.4 Evaluation of line integral with respect to coordinates

Suppose that AB is a smooth curve in the plain

$$x = x(t),\ y = y(t)$$

and the functions $X(x,y)$ and $Y(x,y)$ are continuous on AB. Let at the point A the parameter $t = \alpha$ and at the point B $t = \beta$.

Theorem 1. If the functions $X(x, y)$ and $Y(x, y)$ are continuous on the smooth curve AB, then

$$\int_{AB} X(x,y)dx + Y(x,y)dy = \int_{\alpha}^{\beta} [X(x(t), y(t))\dot{x} + Y(x(t), y(t))\dot{y}]dt \qquad (8.11)$$

Proof. We prove the first half of this equality, i.e.

$$\int_{AB} X(x,y)dx = \int_{\alpha}^{\beta} X(x(t), y(t))\dot{x}dt$$

Denote the function of the parameter $\varphi(t) = X(x(t), y(t))$. The smoothness of the curve AB means that $x(t)$, $y(t)$, $\dot{x}$ are $\dot{y}$ continuous on $[\alpha; \beta]$. By the definition of the line integral with respect to coordinates

$$\int_{AB} X(x,y)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} X(\xi_k, \eta_k)\Delta x_k$$

If the point P_k is related to the parameter t_k, then $x_k = x(t_k)$ and $y_k = y(t_k)$. By Lagrange theorem there exists $\tau_k \in (t_{k-1}; t_k)$ such that

$$\Delta x_k = x_k - x_{k-1} = \dot{x}(\tau_k)(t_k - t_{k-1}) = \dot{x}(\tau_k)\Delta t_k$$

for any $k = 1,\ 2,\ \ldots,\ n$.

In the definition of the line integral $Q_k(\xi_k, \eta_k)$ is a whatever point on the kth subarc, therefore the point related to the value of the parameter τ_k can be chosen, that is $\xi_k = x(\tau_k)$ and $\eta_k = y(\tau_k)$. According to our notation $X(\xi_k, \eta_k) = \varphi(\tau_k)$ and

$$\int_{AB} X(x,y)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \varphi(\tau_k)\dot{x}(\tau_k)\Delta t_k$$

The inverse function of the continuous function $x = x(t)$ is continuous. Thus, $\Delta x_k \to 0$ yields $\Delta t_k \to 0$ and also $\max \Delta x_k \to 0$ yields $\max \Delta t_k \to 0$ and

$$\int_{AB} X(x,y)dx = \lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} \varphi(\tau_k)\dot{x}(\tau_k)\Delta t_k$$

The limit obtained is the limit of the integral sum of the function $\varphi(t)\dot{x}$ over the interval $[\alpha; \beta]$. Consequently, the limit equals to the definite integral

$$\int_{\alpha}^{\beta} \varphi(t)\dot{x}dt$$

According to the meaning of $\varphi(t)$

$$\int\limits_{AB} X(x,y)dx = \int\limits_{\alpha}^{\beta} X(x(t),y(t))\dot{x}dt$$

which is we wanted to prove.

In three dimensional case there holds the similar theorem. Suppose that on the line AB

$$x = x(t),\ \ y = y(t),\ \ z = z(t)$$

there is defined a vector function $\overrightarrow{F}(x,y,z) = X(x,y,z), Y(x,y,z), Z(x,y,z)$. Suppose again that at the point A the parameter $t = \alpha$ and at the point B $t = \beta$.

Theorem 2. If the functions $X(x,y,z)$, $Y(x,y,z)$ and $Z(x,y,z)$ are continuous on the smooth line AB, then

$$\int\limits_{AB} X(x,y,z)dx + Y(x,y,z)dy + Z(x,y,z)dz$$
$$= \int\limits_{\alpha}^{\beta} [X(x(t),y(t),z(t))\dot{x} + Y(x(t),y(t),z(t))\dot{y} + Z(x(t),y(t),z(t))\dot{z}]dt \tag{8.12}$$

Conclusion. Suppose the plain curve AB is the graph of the function $y = y(x)$ given explicitly and at the point A $x = a$ and at B $x = b$. Treating the variable x as a parameter, we obtain $\dot{x} = 1$, $\dot{y} = y'$ and by the formula (8.11)

$$\int\limits_{AB} X(x,y)dx + Y(x,y)dy = \int\limits_{a}^{b} [X(x,y(x)) + Y(x,y(x))y']dx \tag{8.13}$$

Remark. Sometimes (especially for vertical lines) it is necessary to consider y as the independent variable and x as the function $x = x(y)$. Changing the roles of the variables x and y, we get

$$\int\limits_{AB} X(x,y)dx + Y(x,y)dy = \int\limits_{a}^{b} [X(x(y),y)x' + Y(x(y),y)]dy \tag{8.14}$$

A curve L is called *closed* if its initial and final points are the same point. For example a circle is a closed curve. A curve L is called *simple* if it doesn't cross itself. A circle is a simple curve while a figure ∞ type curve is not simple. If L is not a smooth curve, but can be broken into a finite number of smooth curves, then we say that L is *piecewise smooth*. The line integral over the piecewise smooth closed simple curve L is often denoted

$$\oint\limits_L X(x,y)dx + Y(x,y)dy$$

The *positive orientation* of the closed curve L is that as we traverse the curve following the positive orientation the region D bounded by L must always be on the left.

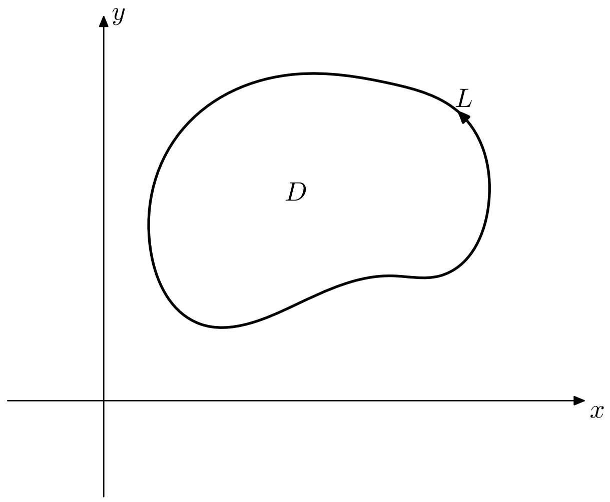

Figure 8.2. The positive orientation

Example 1. Compute $\int\limits_{AB} x\cos y dx - y\sin x dy$ over the straight line from $A(0;0)$ to $B(\pi;2\pi)$.

The direction vector of the line is $\overrightarrow{AB} = (\pi;2\pi)$ and the parametric equations

$$x = \pi t$$
$$y = 2\pi t,$$

At the point A the parameter $t = 0$ and at the point B $t = 1$. To apply the formula (8.11) we find $\dot{x} = \pi$ and $\dot{y} = 2\pi$. By the formula

$$\int\limits_{AB} x\cos y dx - y \sin x dy = \int\limits_0^1 (\pi t \cos 2\pi t \cdot \pi - 2\pi t \sin \pi t \cdot 2\pi) dt$$

$$= \pi^2 \int\limits_0^1 [t(\cos 2\pi t - 4\sin \pi t)] dt = \dots$$

The integral obtained we integrate by parts, taking

$$u = t, \qquad dv = \cos 2\pi t - 4\sin \pi t$$

Then

$$du = dt, \qquad v = \frac{1}{2\pi}\sin 2\pi t + \frac{4}{\pi}\cos \pi t$$

and

$$\dots = \pi^2 \left[t\left(\frac{1}{2\pi}\sin 2\pi t + \frac{4}{\pi}\cos \pi t\right)\Bigg|_0^1 - \int\limits_0^1 \left(\frac{1}{2\pi}\sin 2\pi t + \frac{4}{\pi}\cos \pi t\right) dt\right]$$

$$= \pi^2 \left[-\frac{4}{\pi} + \left(\frac{1}{4\pi^2}\cos 2\pi t - \frac{4}{\pi^2}\sin \pi t\right)\Bigg|_0^1 \right] = -4\pi$$

Example 2. Compute $\oint\limits_L (x^2 + y)dx + xydy$, where L is the positively oriented triangle OAB with vertices $O(0;0)$, $A(2;1)$ and $B(0;1)$.

The triangle is sketched in Figure 7.3. Notice that the triangle is a simple closed piecewise smooth curve, because it consists of three smooth lines. By Property 1

$$\oint\limits_L (x^2+y)dx+xydy = \int\limits_{OA} (x^2+y)dx+xydy + \int\limits_{AB} (x^2+y)dx+xydy + \int\limits_{BO} (x^2+y)dx+xydy$$

By Property 2 the direction is important. Compute all three line integrals. The side OA has the equation $y = \dfrac{x}{2}$, $0 \le x \le 2$ and $y' = \dfrac{1}{2}$. By the formula (8.13)

$$\int\limits_{OA} (x^2 + y)dx + xydy = \int\limits_0^2 \left(x^2 + \frac{x}{2} + x \cdot \frac{x}{2} \cdot \frac{1}{2}\right) dx = \int\limits_0^2 \left(\frac{5x^2}{4} + \frac{x}{2}\right) dx$$

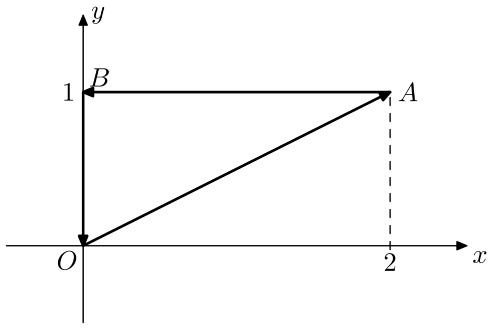

Figure 8.3. The positively oriented triangle OAB

The side AB has the equation $y = 1$, hence, $y' = 0$. At the initial point A $x = 2$ and at the end point B $x = 0$. Thus, by (8.13)

$$\int\limits_{AB} (x^2 + y)dx + xydy = \int\limits_2^0 (x^2 + 1 + x \cdot 1 \cdot 0)dx = \int\limits_2^0 (x^2 + 1)dx$$

The third side BO of the triangle is the vertical line $x = 0$, hence, $x' = 0$. At the point B $y = 1$ and at the point O $y = 0$. To compute the third line integral we use the formula (8.14)

$$\int\limits_{BO} (x^2 + y)dx + xydy = \int\limits_1^0 [(0 + y) \cdot 0 + 0 \cdot y]dy = 0$$

Therefore,

$$\oint\limits_L (x^2 + y)dx + xydy = \int\limits_0^2 \left(\frac{5x^2}{4} + \frac{x}{2}\right) dx + \int\limits_2^0 (x^2 + 1)dx$$

Changing the limits in the last integral gives

$$\oint\limits_L (x^2 + y)dx + xydy = \int\limits_0^2 \left(\frac{5x^2}{4} + \frac{x}{2} - x^2 - 1\right) dx$$

$$= \int\limits_0^2 \left(\frac{x^2}{4} + \frac{x}{2} - 1\right) dx = \left(\frac{x^3}{12} + \frac{x^2}{4} - x\right)\Bigg|_0^2 = \frac{2}{3} + 1 - 2 = -\frac{1}{3}$$

We shall return to the last example once more.

8.5 Green's formula

In this subsection we are going to investigate the relationship between certain kinds of line integrals (on closed curves) and double integrals. Suppose the functions $X(x, y)$ and $Y(x, y)$ are defined on the simple closed curve L and in the region D enclosed by this curve.

Theorem (Green's formula). If the functions $X(x, y)$ and $Y(x, y)$ are continuous on the closed simple smooth curve L, the partial derivatives $\dfrac{\partial Y}{\partial x}$ and $\dfrac{\partial X}{\partial y}$ are continuous in the regular region D and L is positively oriented, then

$$\oint_L X(x, y)dx + Y(x, y)dy = \iint_D \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dxdy \qquad (8.15)$$

Proof. Suppose that L is the closed simple smooth positively oriented curve $AEBFA$, where A is the leftmost and B the rightmost point on the curve. Assume that the lower part AEB of the curve L is the graph of the function $y = \varphi_1(x)$ and the upper part AFB the graph of the function $y = \varphi_2(x)$.

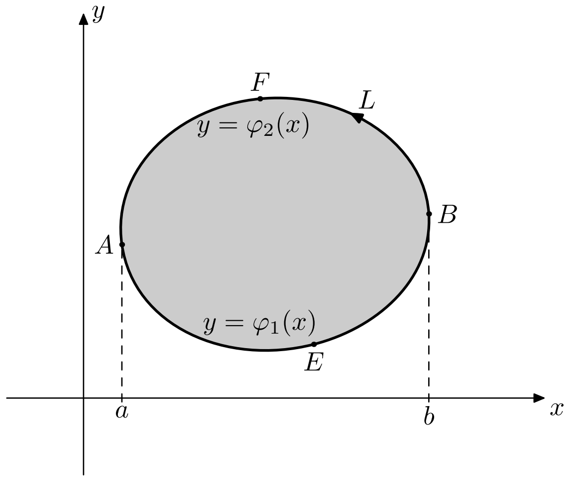

Figure 8.4. Positively oriented closed curve L and the region D, enclosed by the curve

We shall prove the first part of this formula

$$\oint_L X(x, y)dx = -\iint_D \frac{\partial X}{\partial y} dxdy$$

The left side of this formula is by Properties 1 and 2 of the line integral with respect to coordinates.

$$\oint_L X(x,y)dx = \int_{AEB} X(x,y)dx + \int_{BFA} X(x,y)dx = \int_{AEB} X(x,y)dx - \int_{AFB} X(x,y)dx$$

Using the formula (8.13) gives

$$\oint_L X(x,y)dx = \int_a^b X(x,\varphi_1(x))dx - \int_a^b X(x,\varphi_2(x))dx = -\int_a^b [X(x,\varphi_2(x)) - X(x,\varphi_1(x))]dx \tag{8.16}$$

Since the region of integration D is regular, it is determined by inequalities $a \leq x \leq b$ and $\varphi_1(x) \leq y \leq \varphi_2(x)$, thus,

$$\iint_D \frac{\partial X}{\partial y} dxdy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial X}{\partial y} dy$$

In the inside integral the function of two variables $X(x,y)$ is first differentiated with respect to the variable y and then integrated with respect to the same variable. The result is the same function $X(x,y)$. Hence,

$$\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial X}{\partial y} dy = X(x,y)\Big|_{\varphi_1(x)}^{\varphi_2(x)} = X(x,\varphi_2(x)) - X(x,\varphi_1(x))$$

which gives

$$\iint_D \frac{\partial X}{\partial y} dxdy = \int_a^b [X(x,\varphi_2(x)) - X(x,\varphi_1(x))]dx \tag{8.17}$$

The results (8.16) and (8.17) differ only by sign, therefore,

$$\oint_L X(x,y)dx = -\iint_D \frac{\partial X}{\partial y} dxdy \tag{8.18}$$

In similar way we can prove that

$$\oint_L Y(x,y)dy = \iint_D \frac{\partial Y}{\partial x} dxdy \tag{8.19}$$

Adding two equalities (8.18) and (8.19), we obtain (8.15).

Remark. If the curve is piecewise smooth, the proof is nearly the same.

Example. Let us compute the line integral

$$\oint_L (x^2+y)dx + xydy$$

given in Example 2 of the previous subsection once more, using the Green's formula.

Here $X(x,y) = x^2 + y$ and $Y(x,y) = xy$. To apply the Green's formula (8.15) we find $\frac{\partial Y}{\partial x} = y$ and $\frac{\partial X}{\partial y} = 1$. Let D be the region bounded by L. By the formula (8.15)

$$\oint_L (x^2+y)dx + xydy = \iint_D (y-1)dxdy$$

Using Figure 7.3, we determine the limits of integration $0 \le x \le 2$ and $\frac{x}{2} \le y \le 1$. Hence,

$$\oint_L (x^2+y)dx + xydy = \int_0^2 dx \int_{\frac{x}{2}}^1 (y-1)dy$$

Find the inside integral

$$\int_{\frac{x}{2}}^1 (y-1)dy = \int_{\frac{x}{2}}^1 (y-1)d(y-1) = \frac{(y-1)^2}{2}\Bigg|_{\frac{x}{2}}^1 = -\frac{\left(\frac{x}{2}-1\right)^2}{2} = -\frac{(x-2)^2}{8}$$

and the outside integral

$$\int_0^2 \left[-\frac{(x-2)^2}{8}\right] dx = -\frac{1}{8}\int_0^2 (x-2)^2 d(x-2) = -\frac{1}{8}\frac{(x-2)^3}{3}\Bigg|_0^2 = \frac{1}{8}\frac{(-2)^3}{3} = -\frac{1}{3}$$

8.6 Path independent line integral

In this subsection we find out in what conditions the line integral

$$\int_{AB} X(x,y)dx + Y(x,y)dy \tag{8.20}$$

depends only on the endpoints A and B of the line but not on the path of integration.

Assume that in the region D containing the points A and B the functions $X(x,y)$ and $Y(x,y)$ and the partial derivatives $\dfrac{\partial X}{\partial y}$ and $\dfrac{\partial Y}{\partial x}$ are continuous. Let's choose two whatever curves AEB and AFB in the region D joining the points A and B.

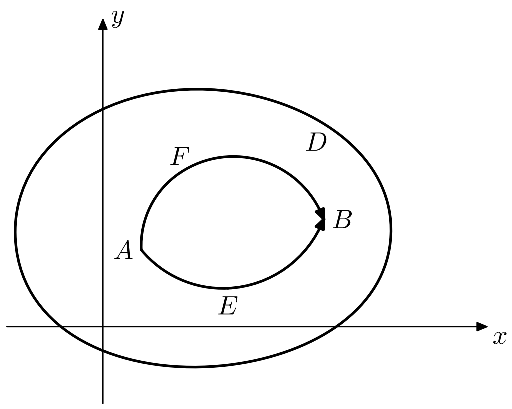

Figure 8.5. Two curves between A and B

So, we want to know in which conditions for any curves AEB and AFB

$$\int\limits_{AEB} Xdx + Ydy = \int\limits_{AFB} Xdx + Ydy$$

i.e.

$$\int\limits_{AEB} Xdx + Ydy - \int\limits_{AFB} Xdx + Ydy = 0$$

By Property 2 of the line integral with respect to coordinates

$$\int\limits_{AEB} Xdx + Ydy + \int\limits_{BFA} Xdx + Ydy = 0$$

and by Property 1

$$\int\limits_{AEBFA} Xdx + Ydy = 0$$

Denoting the closed curve $AEBFA = L$, we obtain the condition

$$\oint_L Xdx + Ydy = 0 \tag{8.21}$$

This condition we obtain for any curves between any two points A and B in the region D. We shall call the curve joining the points A and B the *path of integration*.

Consequently, if the line integral (8.20) is path independent, then for each closed curve L in the region D there holds (8.21).

On the contrary, suppose that for any closed curve L in the region D there holds (8.21). For any two points A and B in the region D we can choose the closed curve L so that these two points are on the curve $L = AEBFA$. By the condition (8.21)

$$\oint_L Xdx + Ydy = \int_{AEBFA} Xdx + Ydy = 0$$

Property 1 of the line integral with respect to coordinates yields

$$\int_{AEB} Xdx + Ydy + \int_{BFA} Xdx + Ydy = 0$$

and Property 2

$$\int_{AEB} Xdx + Ydy - \int_{AFB} Xdx + Ydy = 0$$

or

$$\int_{AEB} Xdx + Ydy = \int_{AFB} Xdx + Ydy$$

Since L is whatever closed curve passing A and B, we have two random paths of integration AEB and AFB joining the points A and B. Consequently, there holds.

Theorem 1. The line integral (8.20) is path independent in the region D if and only if for any closed curve L in the region D there holds (8.21).

Next, suppose that for every closed curve L in the region D there holds (8.21). By the assumptions made in the beginning of this subsection there holds Green's formula. Denote by Δ the region enclosed by the closed curve L. According to Green's formula (8.15)

$$\iint_{\Delta} \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dxdy = 0$$

Then also

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = 0 \tag{8.22}$$

To prove it assume that at some point $P_0 \in D$

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} > 0 \tag{8.23}$$

Since $\dfrac{\partial Y}{\partial x}$ and $\dfrac{\partial X}{\partial y}$ are continuous, the point P_0 has the neighborhood $U(P_0)$ such that in this neighborhood there holds (8.23). But then

$$\iint\limits_{U(P_0)} \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dxdy > 0$$

and if Γ is the boundary of the neighborhood $U(P_0)$, then by Green's theorem

$$\oint\limits_{\Gamma} Xdx + Ydy > 0$$

which contradicts the assumption. Thus, the condition (8.23) is not valid, therefore, there holds (8.22) or

$$\frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y} \tag{8.24}$$

On the contrary, if there holds (8.24) and L is the closed curve enclosing the region Δ, then

$$\iint\limits_{\Delta} \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dxdy = 0$$

which by Green's formula yields (8.21).

Now Theorem 1 gives us the following theorem.

Theorem 2. The line integral (8.20) is path independent in the region D if and only if in the region D there holds the condition (8.24).

The path independent line integral (8.20) is also denoted by

$$\int\limits_{A}^{B} Xdx + Ydy$$

Example 1. The line integral

$$\int\limits_{A}^{B} (2x\cos y - y^2 \sin x)dx + (2y \cos x - x^2 \sin y)dy$$

is path independent because

$$\frac{\partial}{\partial x}(2y\cos x - x^2\sin y) = -2y\sin x - 2x\sin y$$

and

$$\frac{\partial}{\partial y}(2x\cos y - y^2\sin x) = -2x\sin y - 2y\sin x$$

Example 2. Compute

$$\int\limits_{(0,0)}^{(2,1)} 2xydx + x^2dy$$

This line integral is path independent because

$$\frac{\partial(x^2)}{\partial x} = 2x$$

and

$$\frac{\partial(2xy)}{\partial y} = 2x$$

Thus, we can choose whatever path of integration joining the points $(0;0)$ and $(2;1)$. Let's choose the broken line OBA, where $O(0,0)$, $B(2;0)$ and $A(2;1)$. Usually, choosing the kind of broken line, whose segments are parallel to coordinate axes, gives us the most simple computation.

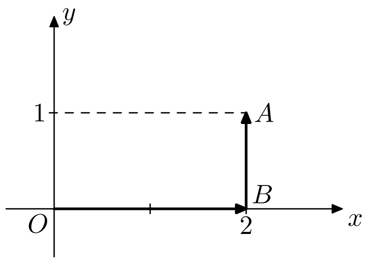

Figure 8.6. The broken line OBA

By Property 1 of the line integral with respect to coordinates

$$\int\limits_{(0,0)}^{(2,1)} 2xydx + x^2dy = \int\limits_{(0,0)}^{(2,0)} 2xydx + x^2dy + \int\limits_{(2,0)}^{(2,1)} 2xydx + x^2dy$$

The equation of the line OB is $y = 0$, which gives $y' = 0$. On the segment OB $0 \leq x \leq 2$ and by the formula (8.13)

$$\int\limits_{(0,0)}^{(2,0)} 2xydx + x^2dy = \int\limits_0^2 (2x \cdot 0 + x^2 \cdot 0)dx = 0$$

The equation of the line BA is $x = 2$, i.e. $x' = 0$. On the segment BA the variable $0 \leq y \leq 1$ and by the formula (8.14)

$$\int\limits_{(2,0)}^{(2,1)} 2xydx + x^2dy = \int\limits_0^1 (4y \cdot 0 + 4)dy = 4$$

Hence,

$$\int\limits_{(0,0)}^{(2,1)} 2xydx + x^2dy = 4$$

If there exists a function of two variables $u(x, y)$ such that the total differential of this function is

$$du = X(x, y)dx + Y(x, y)dy$$

i.e. $X = \dfrac{\partial u}{\partial x}$ and $Y = \dfrac{\partial u}{\partial y}$, then

$$\frac{\partial X}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}$$

and

$$\frac{\partial Y}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}$$

Because of continuity the condition (8.24) holds.

Recall that the vector field $\overrightarrow{F} = (X(x, y), Y(x, y))$ is conservative, if $\overrightarrow{F}$ is the gradient of a scalar field $u(x, y)$ and the function $u(x, y)$ is the potential function of $\overrightarrow{F}$. Then $du = X(x, y)dx + Y(x, y)dy$ is the total differential of $u(x, y)$ and the condition (8.24) holds.

Conclusion 1. For the conservative vector field $\overrightarrow{F} = (X(x, y), Y(x, y))$ the line integral (8.20) is path independent.

Conclusion 2. For the conservative vector field $\overrightarrow{F} = (X(x,y), Y(x,y))$ the line integral over any closed curve L

$$\oint_L X(x,y)dx + Y(x,y)dy = 0$$

Conclusion 3. If $u(x,y)$ is the potential function of the conservative vector field $\overrightarrow{F} = (X(x,y), Y(x,y))$, then

$$\int_A^B X(x,y)dx + Y(x,y)dy = \int_A^B du(x,y) = u(x,y)\Big|_A^B$$

8.7 Surface integral of scalar fields

In mathematical analysis, a surface integral is a generalization of multiple integrals to integration over surfaces. It is like the double integral analog of the line integral. One may integrate over given surface scalar fields and vector fields. Let's start from the integration scalar fields over surface.

Suppose that the function of three variables $f(x,y,z)$ is defined on the surface S in the xyz axes. Choose whatever partition of the surface S into n

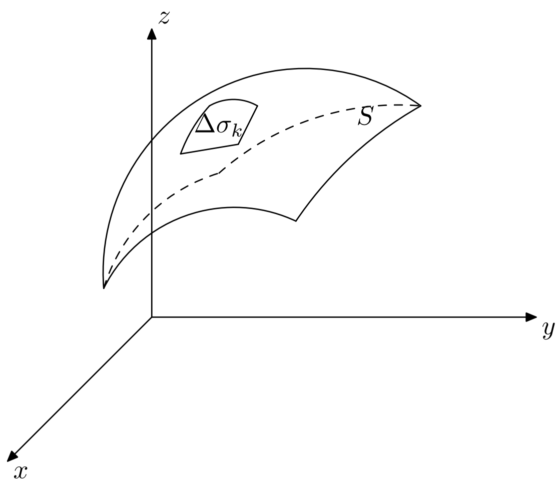

Figure 8.7. The surface S and its subsurface

subsurfaces $\Delta\sigma_k$ $(1 \le k \le n)$, where $\Delta\sigma_k$ denotes the kth subsurface as well as its area.

On any of these subsurfaces we pick a random point $P_k(\xi_k; \eta_k; \zeta_k) \in \Delta\sigma_k$ and find the products

$$f(P_k)\Delta\sigma_k$$

Adding those products, we get the *integral sum* of the function $f(x, y, z)$ *over the surface* S

$$\sum_{k=1}^{n} f(P_k)\Delta\sigma_k$$

The greatest distance between the points on the subsurface is called the diameter of the subsurface diam $\Delta\sigma_k$. Every subsurface has its own diameter. In general those diameters are different because we have the random partition of the surface S. Denote the greatest diameter by λ, i.e.

$$\lambda = \max_{1\le k\le n} \operatorname{diam} \Delta\sigma_k$$

Definition 1. If there exists the limit

$$\lim_{\lambda\to 0} \sum_{k=1}^{n} f(P_k)\Delta\sigma_k$$

and this limit does not depend on the partition of the surface S and does not depend on the choice of points P_k on the subsurfaces, then this limit is called the *surface integral with respect to area of surface* and denoted

$$\iint_S f(x, y, z)d\sigma$$

By Definition 1

$$\iint_S f(x, y, z)d\sigma = \lim_{\lambda\to 0} \sum_{k=1}^{n} f(P_k)\Delta\sigma_k$$

Sometimes the surface integral with respect to area of surface is referred as the *surface integral of the scalar field.* The properties of the surface integral with respect to area of surface are familiar already. While formulating the properties, we use the term "surface integral" and "with respect to area of surface" will be omitted.

Property 1. The surface integral of the sum (difference) of two functions equals to the sum (difference) of surface integrals of these functions:

$$\iint_S [f(x, y, z) \pm g(x, y, z)]d\sigma = \iint_S f(x, y, z)d\sigma \pm \iint_S g(x, y, z)d\sigma$$

Property 2. The constant factor can be taken outside the surface integral, i.e. if c is a constant then

$$\iint\limits_S cf(x,y,z)d\sigma = c\iint\limits_S f(x,y,z)d\sigma$$

Property 3. If the surface is the unit of two surfaces, $S = S_1 \cup S_2$ and S_1 and S_2 have no common interior point, then

$$\iint\limits_S f(x,y,z)d\sigma = \iint\limits_{S_1} f(x,y,z)d\sigma + \iint\limits_{S_2} f(x,y,z)d\sigma$$

Suppose the surface S is the graph of the function of two variables $z = z(x,y)$. Denote by D the projection of the surface S onto xy plane. The surface S is called *smooth* if the function $z(x,y)$ has continuous partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in D.

The following theorem gives the formula to evaluate the surface integral with respect to area of surface.

Theorem. If the function $f(x,y,z)$ is continuous on the smooth surface S and D is the projection of S onto xy plane, then

$$\iint\limits_S f(x,y,z)d\sigma = \iint\limits_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy \quad (8.25)$$

Thus, in order to evaluate a surface integral we will substitute the equation of the surface in for z in the integrand and then add on the factor square root. After that the integral is a standard double integral and by this point we should be able to deal with that.

If the function $f(x,y,z) \equiv 1$ on the surface S, then the formula

$$\iint\limits_S d\sigma = \iint\limits_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy \quad (8.26)$$

gives us the area of the surface S.

Example 1. Evaluate $\iint\limits_S (x^2+y^2+z^2)d\sigma$, if S is the portion of the cone $z = \sqrt{x^2+y^2}$, where $0 \le z \le 1$.

The plane $z = 1$ and the cone $z = \sqrt{x^2+y^2}$ intersect along the circle

$$x^2 + y^2 = 1$$

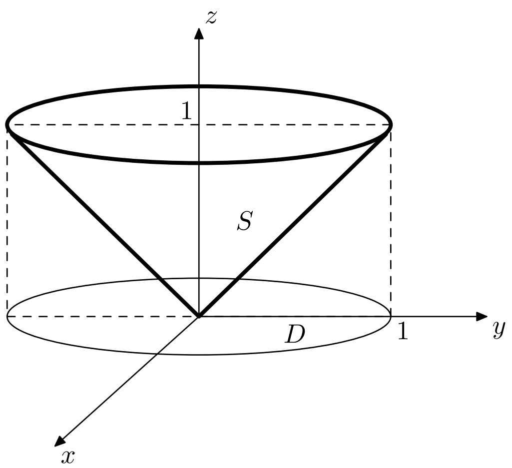

Figure 8.8. The portion of cone in Example 1

The projection of the portion of the cone onto xy plane is the disk $x^2+y^2 \leq 1$. To apply the formula (8.25) we find

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

and

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}} = \sqrt{2}$$

By the formula (8.25)

$$\iint\limits_S (x^2+y^2+z^2)d\sigma = \iint\limits_D (x^2+y^2+x^2+y^2)\sqrt{2}dxdy = 2\sqrt{2}\iint\limits_D (x^2+y^2)dxdy$$

The region of integration D in the double integral obtained is the disk of radius 1 centered at the origin. To compute this double integral we convert it into polar coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. Then $x^2+y^2 = \rho^2$ and $|J| = \rho$.

The region of integration in polar coordinates is determined by inequalities $0 \leq \varphi \leq 2\pi$ and $0 \leq \rho \leq 1$. Hence,

$$2\sqrt{2}\iint\limits_D (x^2+y^2)dxdy = 2\sqrt{2}\int\limits_0^{2\pi} d\varphi \int\limits_0^1 \rho^2\rho d\rho$$

First we compute the inside integral

$$\int_0^1 \rho^3 d\rho = \frac{1}{4}$$

and finally the outside integral

$$2\sqrt{2}\int_0^{2\pi} \frac{1}{4} d\varphi = \frac{\sqrt{2}}{2}\int_0^{2\pi} d\varphi = \pi\sqrt{2}$$

Example 2. Compute the area of the portion of paraboloid of rotation $z = x^2 + y^2$ under the plane $z = 4$.

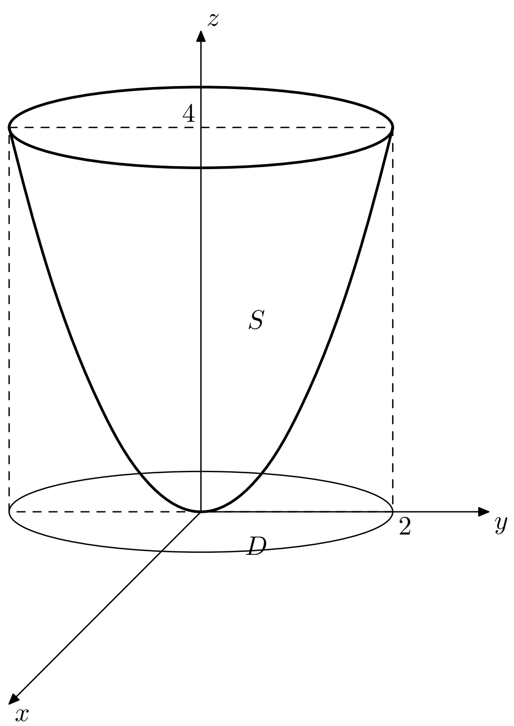

Figure 8.9. The paraboloid of rotation in Example 2

The projection D of the portion of paraboloid of rotation onto xy plane is the disk $x^2 + y^2 \leq 4$ of radius 2 centered at the origin.we find

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y} = 2y$$

and

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

Thus, by the formula (8.26) the area of the portion of paraboloid of rotation is

$$\iint_S d\sigma = \iint_D \sqrt{1 + 4x^2 + 4y^2} dx dy$$

The double integral obtained we convert to polar coordinates $x = \rho \cos \varphi$, $y = \rho \sin \varphi$. Then $1 + 4x^2 + 4y^2 = 1 + 4\rho^2$ and $|J| = \rho$ and the region D is determined by $0 \leq \varphi \leq 2\pi$ and $0 \leq \rho \leq 2$. Therefore,

$$\iint_D \sqrt{1 + 4x^2 + 4y^2} dx dy = \int_0^{2\pi} d\varphi \int_0^2 \sqrt{1 + 4\rho^2} \rho d\rho$$

To find the inside integral we use the equality of differentials $d(1 + 4\rho^2) = 8\rho d\rho$, which gives

$$\begin{aligned} \int_0^2 \sqrt{1 + 4\rho^2} \rho d\rho &= \frac{1}{8} \int_0^2 \sqrt{1 + 4\rho^2} 8\rho d\rho \\ &= \frac{1}{8} \int_0^2 (1 + 4\rho^2)^{\frac{1}{2}} d(1 + 4\rho^2) = \frac{1}{8} \frac{(1 + 4\rho^2)^{\frac{3}{2}}}{\frac{3}{2}} \Bigg|_0^2 \\ &= \frac{1}{12} (1 + 4\rho^2) \sqrt{1 + 4\rho^2} \Bigg|_0^2 = \frac{17\sqrt{17} - 1}{12} \end{aligned}$$

The outside integral, i.e. the area to be computed is

$$\frac{17\sqrt{17} - 1}{12} \int_0^{2\pi} d\varphi = \frac{17\sqrt{17} - 1}{12} \cdot 2\pi = \frac{\pi(17\sqrt{17} - 1)}{6}$$

8.8 Surface integral with respect to coordinates

Suppose that S is a surface in the space and let $Z(x, y, z)$ be a function defined at all points of S. Choose a whatever partition of the surface S into n nonoverlapping subsurfaces $\Delta\sigma_k$ $(1 \leq k \leq n)$. In any of these subsurfaces we pick a random point $P_k(\xi_k;\eta_k;\zeta_k)$ and compute the value of function $Z(P_k)$. Let us denote by Δs_k the projection of $\Delta\sigma_k$ onto xy plane, where Δs_k denotes also the area of this projection. Next we find the products $Z(P_k)\Delta s_k$ and adding these products, we get the sum

$$\sum_{k=1}^{n} Z(P_k)\Delta s_k$$

which is called the integral sum of the function $Z(x,y,z)$ over the projection of surface S onto xy plane. Let $\operatorname{diam}\Delta s_k$ be the diameter of Δs_k. We have a random partition of the surface S, hence the diameters of these projections are different. Denote by λ the greatest diameter of the projections of subsurfaces $\Delta\sigma_k$, i.e.

$$\lambda = \max_{1\le k\le n} \operatorname{diam}\Delta s_k$$

Definition 1. If there exists the limit

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} Z(P_k)\Delta s_k$$

and this limit does not depend on the partition of the surface S and it is independent on the choice of points P_k in the subsurfaces, then this limit is called the *surface integral* of the function $Z(x,y,z)$ *over the projection of the surface* onto xy plane and denoted

$$\iint_S Z(x,y,z)dxdy$$

Thus, by the definition

$$\iint_S Z(x,y,z)dxdy = \lim_{\lambda\to 0}\sum_{k=1}^{n} Z(P_k)\Delta s_k \qquad (8.27)$$

Second, suppose that the function of three variables $Y(x,y,z)$ is defined at all points of the surface S and that $\Delta s'_k$ is the projection of $\Delta\sigma_k$ onto xz plane. Choosing again a random point $P_k\in\Delta\sigma_k$, we find the products $Y(P_k)\Delta s'_k$. The sum of these products

$$\sum_{k=1}^{n} Y(P_k)\Delta s'_k$$

is called the integral sum of the function $Y(x,y,z)$ over the projection of S onto xz plane.

Definition 2. If there exists the limit

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} Y(P_k)\Delta s'_k$$

and this limit does not depend on the partition of the surface S and it is independent on the choice of points P_k in the subsurfaces, then this limit is called the *surface integral* of the function $Y(x, y, z)$ *over the projection of the surface* onto xz plane and denoted

$$\iint_S Y(x, y, z)dxdz$$

By Definition 2

$$\iint_S Y(x, y, z)dxdz = \lim_{\lambda\to 0}\sum_{k=1}^{n} Y(P_k)\Delta s'_k \tag{8.28}$$

Third, suppose that the function of three variables $X(x, y, z)$ is defined at all points of the surface S and $\Delta s''_k$ is the projection of $\Delta\sigma_k$ onto yz plane. We choose again random points $P_k \in \Delta\sigma_k$ and find the products $X(P_k)\Delta s''_k$. The sum

$$\sum_{k=1}^{n} X(P_k)\Delta s''_k$$

is called the integral sum of function $X(x, y, z)$ over the projection of S onto yz plane.

Definition 3. If there exists the limit

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} X(P_k)\Delta s''_k$$

and this limit does not depend on the partition of the surface S and does not depend on the choice of points P_k in the subsurfaces, then this limit is called the *surface integral* of the function $X(x, y, z)$ *over the projection of the surface* onto yz plane and denoted

$$\iint_S X(x, y, z)dydz$$

By Definition 3

$$\iint_S X(x, y, z)dydz = \lim_{\lambda\to 0}\sum_{k=1}^{n} X(P_k)\Delta s''_k \tag{8.29}$$

In general we define the surface integral over the projection of the vector function

$$\overrightarrow{F}(x, y, z) = (X(x, y, z); Y(x, y, z); Z(x, y, z))$$

as

$$\iint_S X(x, y, z)dydz + Y(x, y, z)dxdz + Z(x, y, z)dxdy \qquad (8.30)$$

Remark. Sometimes the surface integral over the projection is also referred as the surface integral of the vector field.

8.9 Evaluation of surface integral over the projection

Consider the evaluation of the surface integral over the projection onto xy plane

$$\iint_S Z(x, y, z)dxdy$$

Suppose that the smooth surface S is a graph of the one-valued function of two variables $z = f(x, y)$. Since the function is one-valued, any line parallel to z axis cuts this surface exactly at one point.

Definition 1. A smooth surface S is said to be *two-sided* or *orientable*, if the normal vector, starting at a point in the surface and moving along any closed curve not crossing the boundary on the surface, is pointing always in the same direction.

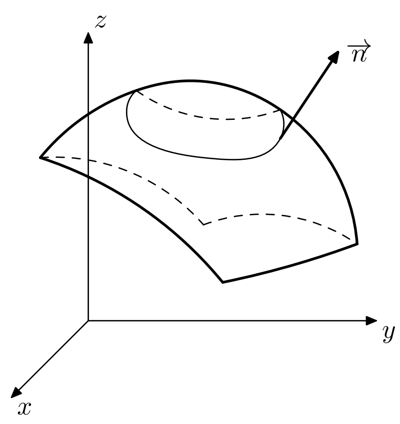

Figure 8.10. Two-sided surface

A well known example of the surface, which cannot be oriented is the *Mbius band.* It consists of a strip of paper with ends joined together to form a loop, but with one end given a half twist before the ends are joined.

For a two-sided surface we differ the upper and the lower side of the surface. The upper side of the surface is the side, where the normal vector forms an acute angle with z axis. The lower side of the surface is the side, where the normal vector forms an obtuse angle with z axis.

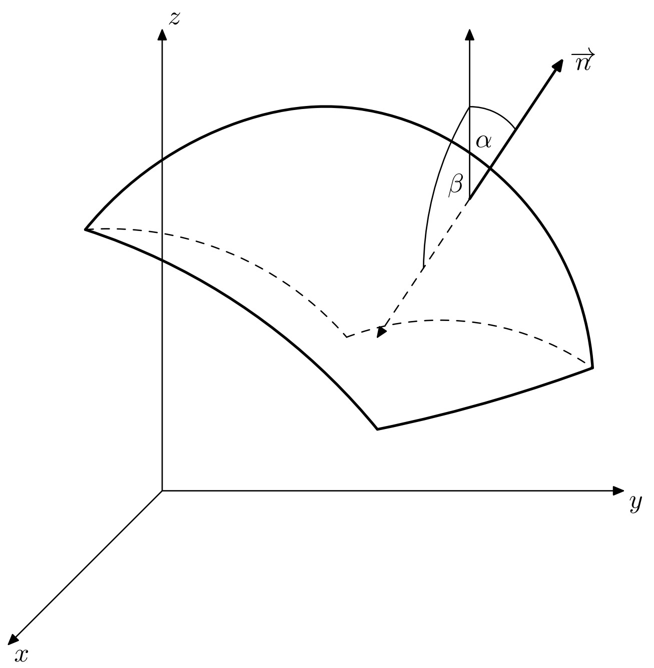

Figure 8.11. The upper and the lower side of surface

The evaluation of the surface integral over the projection depends on the side of the surface over which we integrate. If the function $Z(x, y, z)$ is continuous at any point of the smooth surface $z = f(x, y)$, then the surface integral over the projection onto xy plane is computed by the formula.

$$\iint\limits_S Z(x, y, z)dxdy = \pm \iint\limits_D Z(x, y, f(x, y))dxdy \tag{8.31}$$

On the right side of this formula is a standard double integral, where D denotes the projection of the surface S onto xy plane. Using this formula, we choose the sign "+", if we integrate over the upper side of surface and we choose the sign "−", if we integrate over the lower side of the surface. So, for any problem there has to be said over which side of the surface we need to integrate.

If the function $Y(x,y,z)$ is continuous at any point of the smooth surface $y = g(x,z)$, then the surface integral over the projection onto xz plane is computed by the formula

$$\iint\limits_S Y(x,y,z)dxdz = \pm \iint\limits_{D'} Y(x,g(x,z),z)dxdz \qquad (8.32)$$

In this formula D' denotes the projection of S onto xz plane and the choice of the sign $+$ or $-$ depends on over which side of the surface the integration is carried out (i.e. does the normal of the surface forms with y axis acute or obtuse angle).

If the function $X(x,y,z)$ is continuous at any point of the smooth surface $x = h(y,z)$, then the surface integral over the projection onto yz plane is computed by the formula

$$\iint\limits_S X(x,y,z)dydz = \pm \iint\limits_{D''} X(h(y,z),y,z)dydz \qquad (8.33)$$

Here D'' denotes the projection of S onto yz plane and the choice of the sign $+$ or $-$ depends on over which side of the surface the integration is carried out (i.e. does the normal of the surface forms with x axis acute or obtuse angle).

Example. Compute the surface integral

$$\iint\limits_S z^2 dxdy$$

where S is the upper side of the portion of cone $z = \sqrt{x^2+y^2}$ between the planes $z = 0$ and $z = 1$.

This portion of cone is sketched in Figure 8.8. The projection D onto xy plane of this portion of cone is the disk $x^2 + y^2 \le 1$. Hence by (8.31)

$$\iint\limits_\sigma z^2 dxdy = \iint\limits_D (x^2+y^2)dxdy$$

Since the region of integration is the disk, we convert the double integral into polar coordinates. For this disk $0 \le \varphi \le 2\pi$ and $0 \le \rho \le 1$, thus,

$$\iint\limits_D (x^2+y^2)dxdy = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 \rho^2 \cdot \rho d\rho$$

Now we compute

$$\int_0^1 \rho^3 d\rho = \frac{\rho^4}{4}\bigg|_0^1 = \frac{1}{4}$$

and

$$\int_0^{2\pi} \frac{1}{4} d\varphi = \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$$