

1 Function, limit, continuity

1.1 Function

1.1.1 Notations

The sets of numbers we denote in common way:

$\mathbb{N} = \{0; 1; 2; 3; \dots\}$ - the set of natural numbers;

$\mathbb{Z} = \{0; \pm 1; \pm 2; \pm 3; \dots\}$ - the set of integers;

$\mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{N}, n \neq 0 \right\}$ - the set of rational numbers;

\mathbb{R} - the set of real numbers.

Important subsets of the set of real numbers are *intervals*. Given two real numbers a and b such that $a < b$ the *closed interval* $[a; b]$ is defined as

$$[a; b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

and *open interval* $(a; b)$ is defined as

$$(a; b) = \{x \in \mathbb{R} \mid a < x < b\}$$

In both cases a is called *lower endpoint* of the interval and b is called *upper endpoint* of the interval. The difference of the closed and open interval is that the closed interval contains both endpoints a and b but the open interval contains none of these.

Open intervals are also

$$(a; \infty) = \{x \in \mathbb{R} \mid x > a\}$$

and

$$(-\infty; b) = \{x \in \mathbb{R} \mid x < b\}$$

The *half-open interval* $[a; b)$ is defined as

$$[a; b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

the half-open interval $(a; b]$ is defined as

$$(a; b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

The *half-open interval* $[a; \infty)$ is defined as

$$[a; \infty) = \{x \in \mathbb{R} \mid x \geq a\}$$

the half-open interval $(-\infty; b]$ is defined as

$$(-\infty; b] = \{x \in \mathbb{R} \mid x \leq b\}$$

The *union* of two given sets A and B is denoted $A \cup B$ and is defined as the set of elements that are the elements of A or the elements of B or elements of both sets A and B :

$$A \cup B = \{x|x \in A \text{ or } x \in B\}$$

The *intersection* of two given sets A and B denoted $A \cap B$ and is defined as the set of elements that are the elements of A and the elements of B :

$$A \cap B = \{x|x \in A \text{ and } x \in B\}$$

A domain is the subset of the set of real numbers. Closed and open intervals, half-opened intervals, the unions and intersections of these are domains. We shall denote the domains by capital letters X, Y, Z, \dots

Constant has only one certain value. In calculus first letters of alphabet a, b, c, \dots are usually constants. Well-known constants are

$$\pi = 3,14159265\dots$$

and

$$e = 2,7182818284\dots$$

A *variable* can have any value from some domain. Last letters on alphabet x, y, z, \dots denote variables.

The letters i, j, k, l, m, n are usually used as integer variables. Functions are denoted by f, g, h and by the correspondent Greece letters φ (phi), ψ (psi) χ (hi).

In the course Mathematical analysis the variables are as a rule real variables (if not assumed otherwise). The expression $x \in X$ is read: x is element of X or x is a point in X .

It is customary to use two *quantifiers* - *generality quantifier* \forall and *existential quantifier* \exists . The symbol \forall we read in the text "for all". The symbol \exists we read "there exists". The expression $\forall \varepsilon > 0 \exists \delta > 0$ we read: for every positive ε there exists positive δ .

1.1.2 Function and representation of the function

Definition 1.1. If to each value of the variable x in some domain X is assigned exactly one value of the variable $y \in Y$, then the variable y is called the *function* of the variable x .

Although the function idea places no restriction on the nature of the objects in two sets X and Y , in our course we are primarily interested in

functions where X and Y are subsets of real numbers. Such functions are called real-valued functions of a real variable, or, more briefly, real functions.

In calculus this is expressed as $y = f(x)$, $y = F(x)$, $y = \varphi(x)$ etc. The variable x is called *independent variable* or *argument* the variable y is called *dependent variable*. The symbol f is function, i.e. the rule that organizes the correspondence. Thus - a function is given if there exists a rule that assigns to each value of the independent variable one certain value of the dependent variable.

Functions can be represented as tables, graphs and analytical expressions.

Example 1.1. The table

x	y
-2	3
-1	11
0	15

represents a function according the definition 1.1 because to each value of x from the set $X = \{-2, -1, 0\}$ that contains three elements there is assigned exactly one certain value of y . To the value -2 of x there corresponds the value 3 of y etc.

Second representation of the function is a graph.

Example 1.2. The graph represents a function according to the defi-

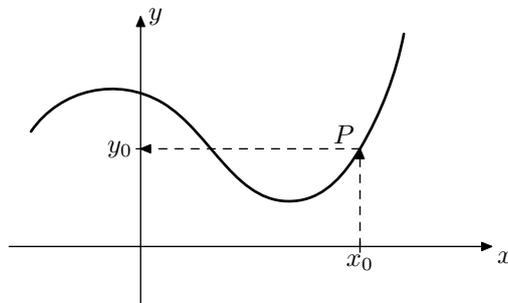


Figure 1.1: Representation of a function as a graph

inition 1.1 because to the arbitrary value x_0 of the independent variable x there corresponds a point P on the graph. The ordinate y_0 of the point P is uniquely determined, consequently to the each value of the argument x there is assigned exactly one certain value of the variable y .

Third representation of the function is analytical expression. Here we distinguish *explicit*, *implicit* and *parametric representation* of a function.

The explicit representation of a function is written as $y = f(x)$ where the right part $f(x)$ is some analytical expression with respect to independent

variable x . All the basic elementary functions are represented explicitly: the quadratic function $y = x^2 - 2x + 3$, trigonometric functions $y = \sin x$, $y = \cos x$, $y = \tan x$ and $y = \cot x$, exponential function $y = a^x$ ($a > 0$ and $a \neq 1$), logarithmic function $y = \log_a x$ ($a > 0$ and $a \neq 1$) etc.

Before we shall deal with the implicit and parametric representation of a function, we need to enlarge the notion of the function. Further we say that the variable y is a function of the variable x if to each value of x , there are assigned two values of the variable y , three values of y , ... , infinitely many values of y . In the first case we say that the function is *two-valued*, in second case we say that the function is *three-valued* and, ... , we say that we have *infinitely many-valued* function. In general we speak about *multiple valued functions*.

Example 1.3. An implicit function is

$$x^2 + y^2 = r^2$$

where r is a positive constant. The graph of this function is the circle with radius r and center at the origin. To give the explicit form of this function

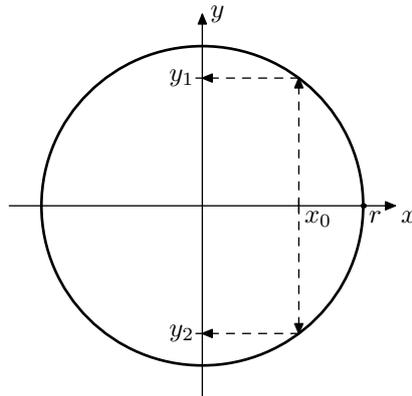


Figure 1.2: The circle $x^2 + y^2 = r^2$

we have to solve this equation for y . First $y^2 = r^2 - x^2$ and the result is $y = \pm\sqrt{r^2 - x^2}$. To each value x_0 of the variable x from the open interval $(-r; r)$ there are assigned two values $y_1 = \sqrt{r^2 - x_0^2}$ and $y_2 = -\sqrt{r^2 - x_0^2}$ of the variable y (see figure). Thus, this is *two-valued* function. The explicit functions $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$ are the functions according to the definition 1.1. We say that these are the unique branches of the two-valued function given implicitly. The graph of the function $x^2 + y^2 = r^2$ given implicitly is the whole circle, the graph of the function $y = \sqrt{r^2 - x^2}$

is upper half of the circle and the graph of the function $y = -\sqrt{r^2 - x^2}$ is lower half of the circle.

Another convenient way to deal with the multi-valued functions (but also with functions satisfying definition 1.1) is *parametric representation*. The parametric representation of a function is a pair of equations

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

where t is called *parameter*. We can give parametric representation to any function. The parametric representation of the square function $y = x^2$ is

$$\begin{cases} x = t \\ y = t^2 \end{cases}$$

but, of course, this is unnecessary.

The parametric representation of the function $x^2 + y^2 = r^2$ is

$$\begin{cases} x = r \cos t \\ y = r \sin t \end{cases}$$

The parameter in this representation is the angle t shown in figure.

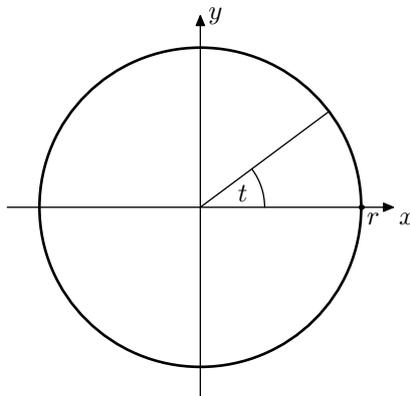


Figure 1.3: The meaning of parameter t

There is no need to use parametric representation of the function $y = x^2$ but we often use the parametric representation of the function $x^2 + y^2 = r^2$.

There are a lot of functions for which the parametric representation is the only reasonable representation.

Example 1.4. Let us consider the function

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$$

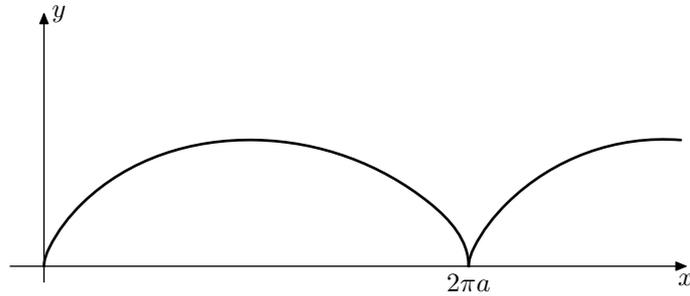


Figure 1.4: Cycloid

The graph of this function is a cyclic line called *cycloid*. A cycloid is the curve traced by one fixed point (with initial position at the origin) of the circle of radius a , if this circle rolls along the x -axis. The parameter t is the angle of rotation of this circle with respect to its initial position. If the circle has made half of rotation, i.e. $t = \pi$, then $x = a(\pi - \sin \pi) = a\pi$, $y = a(1 - \max \cos \pi) = 2a$ and the fixed point is in its highest position at the height $2a$. If the circle has made one full rotation, i.e. $t = 2\pi$, then $x = a(2\pi - \sin 2\pi) = 2\pi a$, $y = a(1 - \cos 2\pi) = 0$ and the fixed point has hit the x -axis again at the distance from origin that is obviously the circumference of this circle.

Definition 1.2. The domain of the function $y = f(x)$ is the set of all values of x such that $f(x)$ is defined.

The domain of the function is prescribed either by a rule or by explicitly giving its domain in addition to a rule. We shall denote a domain of functions with symbol X .

Example 1.5. The domain of the function

$$f(x) = \begin{cases} x, & \text{for } 0 \leq x \leq 1 \\ 2 - x, & \text{for } 1 < x \leq 2 \end{cases}$$

is closed interval $X = [0; 2]$ because outside of this closed interval the function is not defined. The graph of this function is in Figure 1.5.

Example 1.6. The domain of the function $y = \sqrt{2x - x^2}$ is prescribed by the restriction $2x - x^2 \geq 0$. This inequality holds for all x satisfying the conditions $0 \leq x \leq 2$. Thus, the domain of this function is the closed interval $X = [0; 2]$.

Definition 1.3. The range of the function $y = f(x)$ consists of all values of y that correspond to all values of the independent variable x from the domain of the function. The range of the function we shall denote by Y .

Example 1.7. Let us determine the range of the function given in example 1.6. Underneath the radical sign there is quadratic function $2x - x^2$

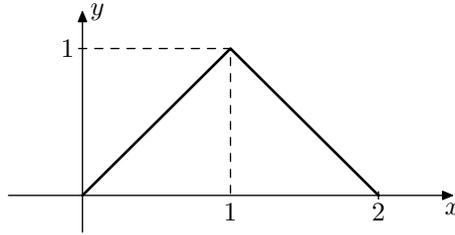


Figure 1.5: The graph of the function of example 1.5

whose graph is a downward opening parabola. At the endpoints of the domain $X = [0; 2]$ the value of the function is 0 and this is the least value of y (remind that the square root is always non-negative). The greatest value of the quadratic function is the ordinate of the vertex of the parabola. The abscissa of the vertex is $x_v = \frac{0 + 2}{2} = 1$ and the ordinate of this point is $y_v = 2 \cdot 1 - 1^2 = 1$. The value 1 is the greatest value of the quadratic function underneath the radical sign and as $\sqrt{1} = 1$, the greatest value of given function is also 1. Thus, the range of the function $Y = [0; 1]$.

1.1.3 Classification of the functions

The functions are classified due to some certain common properties, e.g. symmetric properties, the repetition of the values of functions etc.

Every function can either be classified as an even function, an odd function, or neither.

Definition 1.4. The function $y = f(x)$ is called *even function*, if $\forall x \in X$, there holds

$$f(-x) = f(x).$$

Even functions are square function $y = x^2$, absolute value function $y = |x|$ and cosine function $y = \cos x$. The graphs of the first and second functions are drawn in Figure 1.6 and the graph of the third function in Figure 1.7. If the graph of the even function passes a point $(x; f(x))$, then according to the definition of an even function it also has to pass the point $(-x; f(x))$. These two points in the coordinate plane are symmetrical with respect to y -axis. Consequently, the graph of any even function is symmetrical with respect to y -axis.

Definition 1.5. The function $y = f(x)$ is called *odd function*, if $\forall x \in X$, there holds

$$f(-x) = -f(x).$$

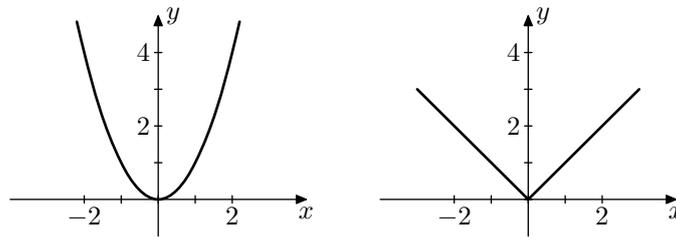


Figure 1.6: the functions $y = x^2$ and $y = |x|$

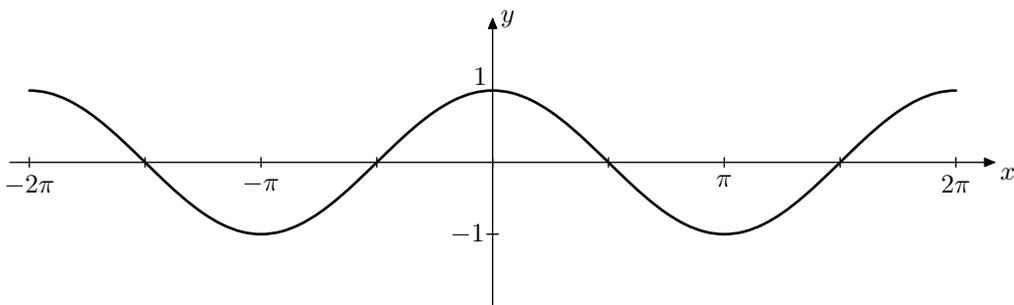


Figure 1.7: the cosine function $y = \cos x$

Odd functions are the cube function $y = x^3$, the sine function $y = \sin x$ and the tangent function $y = \tan x$. The graphs of these functions are drawn in Figures 1.8, 1.9 and 1.10 respectively.

If the graph of the odd function passes a point $(x; f(x))$, then according the definition it also has to pass the point $(-x; -f(x))$. These two points in the coordinate plane are symmetrical with respect to origin. Thus, the graph of any odd function is symmetrical with respect to origin.

Example 1.8. Let us make sure, is $y = \ln \frac{1+x}{1-x}$ even or odd function.

Let $f(x) = \ln \frac{1+x}{1-x}$ and find

$$f(-x) = \ln \frac{1-x}{1-(-x)} = \ln \left(\frac{1+x}{1-x} \right)^{-1} = -\ln \frac{1+x}{1-x} = -f(x)$$

We have got that $\forall x \in X$, there holds $f(-x) = -f(x)$, that means the function is odd.

The following statements are true.

- The product of two even functions is even;

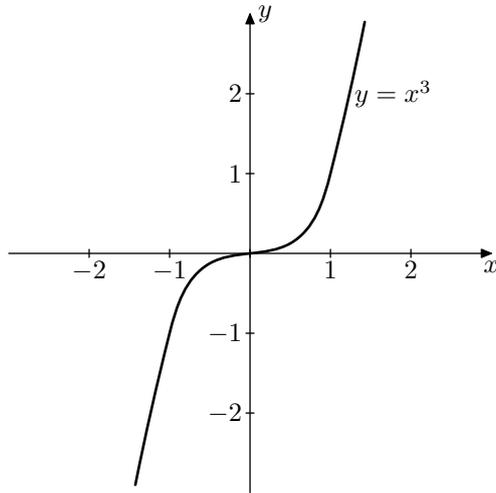


Figure 1.8: the cube function $y = x^3$

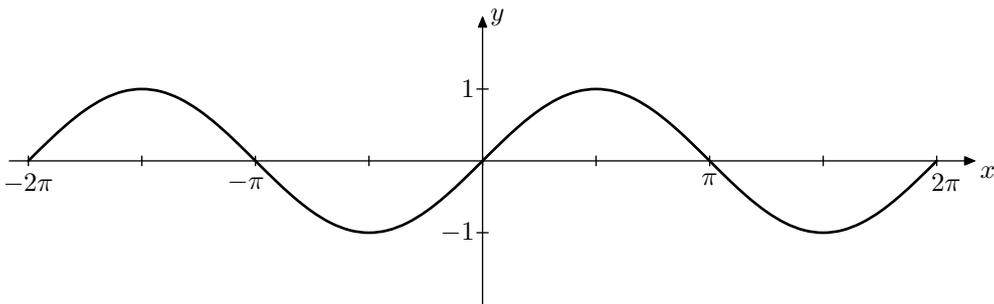


Figure 1.9: the sine function $y = \sin x$

- the product of two odd functions is even;
- the product of an even function and an odd function is odd.

Let us prove the second statement of these three. Consider two odd functions $f(x)$ and $g(x)$. Let $h(x) = f(x)g(x)$ be the product of these two functions and find $h(-x) = f(-x)g(-x) = -f(x) \cdot [-g(x)] = f(x)g(x)$, that means, the product $h(x)$ is indeed even.

Example 1.9. Let us consider the function $y = x \ln \frac{1+x}{1-x}$.

The function $f(x) = x$ is odd, according to the example 1.8 the function $g(x) = \ln \frac{1+x}{1-x}$ is also odd, consequently the product of these two functions is even.

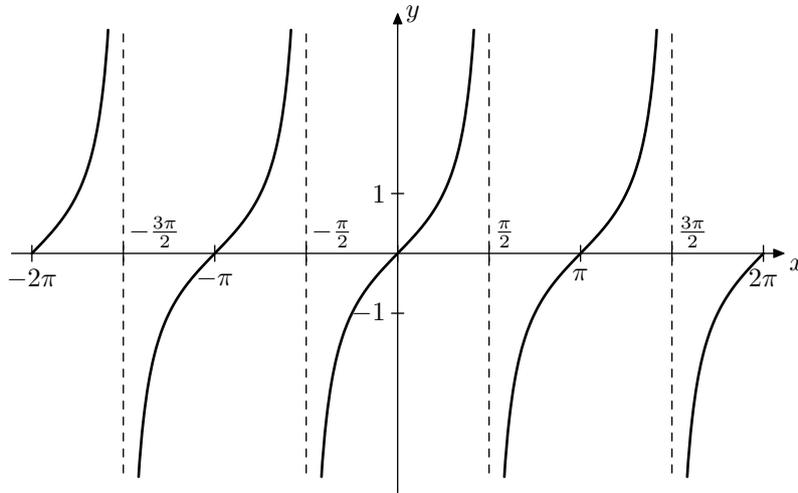


Figure 1.10: the tangent function $y = \tan x$

Definition 1.6. The function $y = f(x)$ is called *periodic*, if there exists real $T \neq 0$ such that $\forall x \in X$, there holds

$$f(x + T) = f(x).$$

Here we assume that $x + T \in X$.

The least positive real number (if it exists) satisfying this condition is called the *period* of the function.

According to the first part of this definition, the sine function is periodic because the real number T in this definition can be 4π , 10π , -6π etc. The least positive such a real number is 2π , which by the second part of the definition is the period of sine function.

The period of the cosine function is also 2π , the period of the tangent function is π .

Trigonometric functions are not the only periodic functions. Actually we can define infinitely many periodic functions with any real period. Let us define the "sawtooth" function

$$f(x) = \begin{cases} x - n, & \text{if } n \leq x < n + 1 \\ 0, & \text{if } x < n \text{ or } x \geq n + 1 \end{cases}$$

where n is whatever integer.

If in this definition $n = 0$, then $f(x) = x$ on the half-open interval $[0; 1)$ and $f(x) = 0$ outside of this half-open interval. If $n = 1$, then $f(x) = x - 1$ on the half-open interval $[1; 2)$ and 0 outside of this half-open interval. If $n = 2$, then $f(x) = x - 2$ on the half-open interval $[2; 3)$ and 0 outside of this

half-open interval. If $n = -1$, then $f(x) = x + 1$ on the half-open interval $[-1; 0)$ and 0 outside of this half-open interval.

The graph of this function is drawn in Figure 1.11, where it has been drawn only the parts of the graph for discussed values of n .

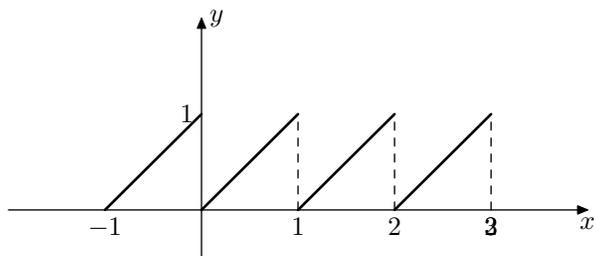


Figure 1.11: the "sawtooth" function

This function is periodic and its period is $T = 1$.

Definition 1.7. The function $y = f(x)$ is called *increasing* on an interval $(a; b)$, if for any values of $x_1, x_2 \in (a; b)$ such that $x_1 < x_2$, there holds $f(x_1) < f(x_2)$.

Sometimes a function with this property is called *strictly increasing*

Thus, the function is increasing on an interval $(a; b)$, if taking two different values of the independent variable x of this interval, the function relates the greater value of x with the greater value of y .

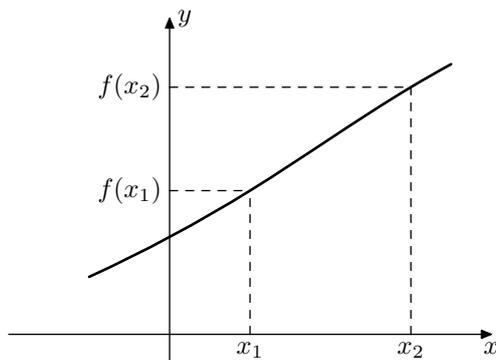


Figure 1.12: the increasing function

Definition 1.8. The function $y = f(x)$ is called *monotonically increasing* on an interval $(a; b)$, if for any values of $x_1, x_2 \in (a; b)$ such that $x_1 < x_2$, there holds $f(x_1) \leq f(x_2)$.

Definition 1.9. The function $y = f(x)$ is called *decreasing* on an interval $(a; b)$, if for any values of $x_1, x_2 \in (a; b)$ such that $x_1 < x_2$, there holds $f(x_1) > f(x_2)$.

A function with this property is also called *strictly decreasing*

Thus, the decreasing function on an interval relates the greater value of x with the less value of y .

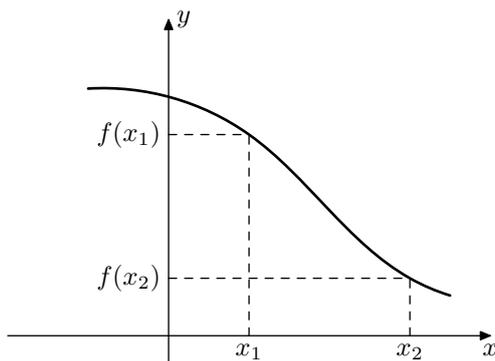


Figure 1.13: the decreasing function

Definition 1.10. The function $y = f(x)$ is called *monotonically decreasing* on an interval $(a; b)$, if for any values of $x_1, x_2 \in (a; b)$ such that $x_1 < x_2$, there holds $f(x_1) \geq f(x_2)$.

The constant function is simultaneously monotonically increasing and monotonically decreasing.

1.1.4 Inverse function

The table in example 1.1 relates each value of x with exactly one value of y . But as well it relates each value of y with exactly one value of x . The variable y in this table we can treat as independent variable and the variable x as dependent variable.

Example 1.10. The function, given explicitly $y = \frac{x}{x+1}$ relates each value of x ($x \neq -1$) with one value of y . Solving this equation for x , we have $x = \frac{y}{1-y}$, i.e. a function that relates each value of y ($y \neq 1$) with one value of x . It turns out that we have two functions: the given function, where y is the function of the variable x and another function, where x is the function of the variable y .

Consequently, with every rule (table, graph, analytic expression) there are defined two functions. The second function is called *inverse function* of

the given function. We shall denote inverse of the function $y = f(x)$ by $x = f^{-1}(y)$. Here -1 means a superscript, not exponent.

Let us find some points on the graph of the function $y = \frac{x}{x+1}$. When $x = 0$, then $y = 0$; when $x = 1$, then $y = \frac{1}{2}$; when $x = -\frac{1}{2}$, then $y = -1$; when $x = -\frac{3}{2}$, then $y = 3$; when $x = -2$, then $y = 2$. So, the graph of this function passes through the points $(0; 0)$, $(1; \frac{1}{2})$, $(-\frac{1}{2}; -1)$, $(-\frac{3}{2}; 3)$ and $(-2; 2)$. As well we find some points on the graph of the inverse function $x = \frac{y}{1-y}$. When $y = 0$, then $x = 0$; when $y = \frac{1}{2}$, then $x = 1$; when $y = -1$, then $x = -\frac{1}{2}$; when $y = 3$, then $x = -\frac{3}{2}$; when $y = 2$, then $x = -2$. Thus, the graph of inverse function passes through the same points as the graph of the given function.

This is a common rule: if $y = f(x)$ is the given function and $x = f^{-1}(y)$ its inverse function the graphs of these two functions coincide.

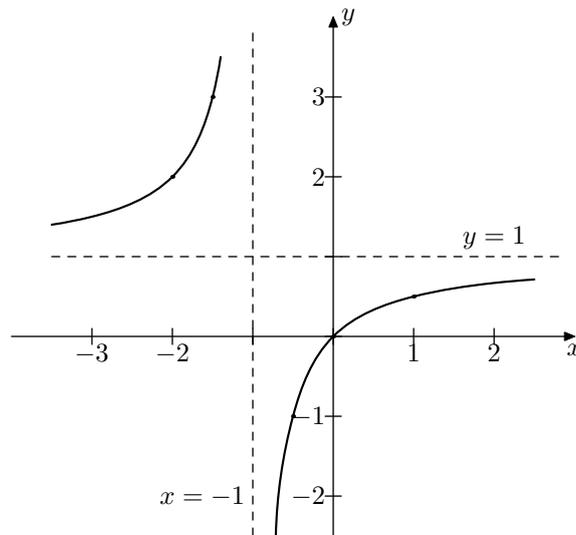


Figure 1.14: the graphs of the functions $y = \frac{x}{x+1}$ and $x = \frac{y}{1-y}$

The graph in Figure 1.14 is a graph of both functions $y = \frac{x}{x+1}$ and $x = \frac{y}{1-y}$. The dots are the points we have found in the previous evaluation.

It is common in the expression of inverse function to exchange x and y , so that independent variable is denoted by x and dependent variable by y

again. In this notation the inverse function of the function $y = f(x)$ is

$$y = f^{-1}(x)$$

If we denote the independent variable of the inverse function of the function $y = \frac{x}{x+1}$ by x , then we have $y = \frac{x}{1-x}$.

We have exchanged the roles of x and y . If the graph of the given function passes through the point $(x; y)$, then the graph of the inverse function passes through the point $(y; x)$. The second point is a reflection of the first point with respect to straight line $y = x$ and vice versa. The straight line $y = x$ is the bisectrix of the 1st and 3rd quadrants of the coordinate plane.

Consequently: the graph of the inverse function $y = f^{-1}(x)$ we can draw by the reflection of the graph of the function $y = f(x)$ with respect to the straight line $y = x$. Let us have some examples.

Example 1.11. Let a given function be the square function $y = x^2$ the graph of whose is in Figure 1.6.

Solving this equation for x , we have double-valued function (but not a function in the sense of definition 1.1) $x = \pm\sqrt{y}$ and after exchanging x and y the inverse function of the square function is $y = \pm\sqrt{x}$. The graph of this inverse function is the reflection of the graph of $y = x^2$ about the line $y = x$ (see Figure 1.15).

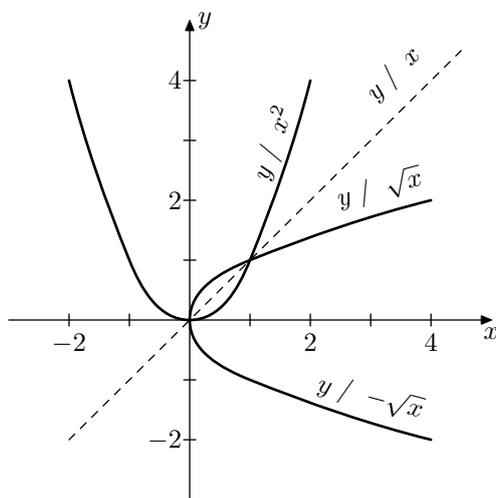


Figure 1.15: square function $y = x^2$ and its inverse function

Example 1.12. Let the given function be the exponential function $y = 2^x$. Here $x = \log_2 y$ and after exchanging x and y we have $y = \log_2 x$. The

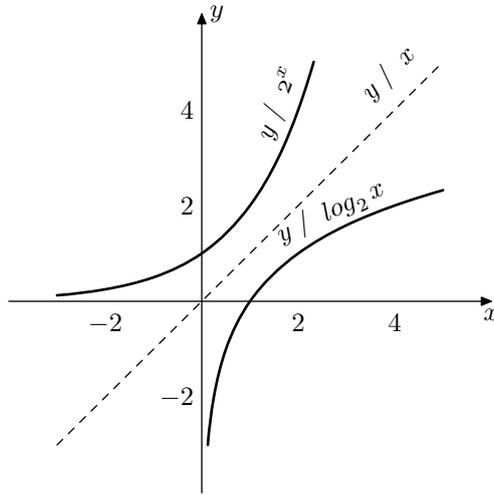


Figure 1.16: the function $y = 2^x$ and its inverse function $y = \log_2 x$

inverse function of the exponential function is the logarithmic function (to the same base).

Taking a look to the graph of sine function $y = \sin x$, whose domain is the set of all real numbers $(-\infty; \infty)$ (Figure 1.9), we notice that to each value of y there correspond infinitely many values of x . Thus, the inverse function of the sine function is infinitely many-valued function. To obtain an uniquely determined function (i.e. the function due to definition 1.1) we impose the restriction $x \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$. The graph of sine function with restriction imposed is drawn in Figure 1.17.

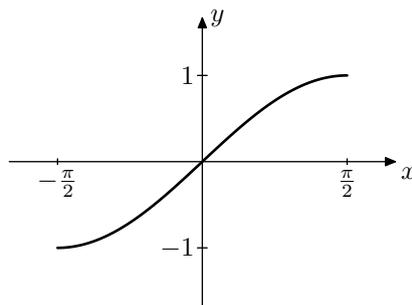


Figure 1.17: the function $y = \sin x$ on the closed interval $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

Now with each value of the variable $y \in [-1; 1]$ is associated one and only one value of the variable x . This function is denoted $x = \arcsin y$ and this is

called *arc sine function*. This is also referred as the principal branch of the inverse function of sine function. Exchanging x and y we have the inverse function

$$y = \arcsin x$$

of the sine function $y = \sin x$ with restriction $y \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$. The domain of this function is $X = [-1; 1]$ and the range $Y = \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$.

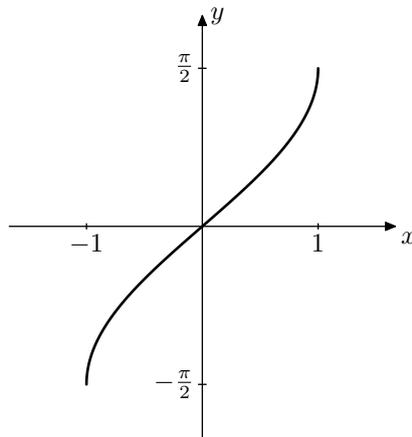


Figure 1.18: the function $y = \arcsin x$

Notice, that the graph of $y = \arcsin x$ is the reflection of the graph of the function $y = \sin x$, $x \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$ about the line $y = x$.

If we solve the equation $y = \sin x$ for x in terms of $y \in [-1; 1]$, the result is

$$x = (-1)^n \arcsin y + n\pi, \quad n \in \mathbb{Z}$$

Exchanging the notation, we obtain infinitely many-valued function

$$y = (-1)^n \arcsin x + n\pi, \quad n \in \mathbb{Z}$$

which is denoted $y = \text{Arcsin } x$

Let the domain of the cosine function be restricted to $[0; \pi]$ (Figure 1.20).

With each value of $y \in [-1; 1]$ is now associated one value of x . This function is called *arc cosine function* and denoted $x = \arccos y$. Exchanging x and y we obtain

$$y = \arccos x$$

, whose domain is $X = [-1; 1]$ and range $Y = [0; \pi]$.

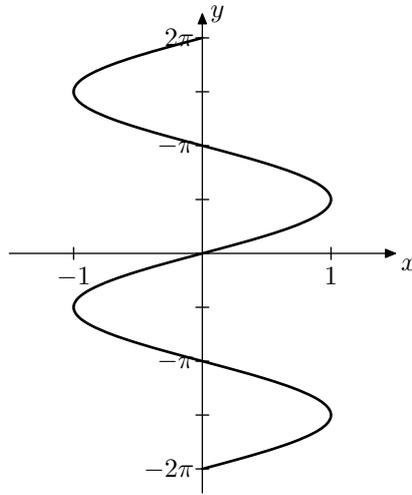


Figure 1.19: the function $y = \text{Arcsin } x$

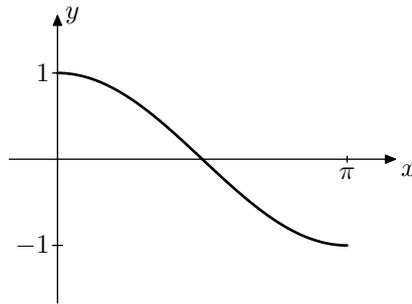


Figure 1.20: the function $y = \cos x$ on the closed interval $[0; \pi]$

For the inverse trigonometric functions $y = \arcsin x$ and $y = \arccos x$ for each $x \in [-1; 1]$ there holds the equality

$$\arcsin x + \arccos x = \frac{\pi}{2}$$

If we solve the equation $y = \cos x$ for x in terms of y , we obtain $x = \pm \arccos y + 2n\pi$, $n \in \mathbb{Z}$. After exchange the notation we have infinitely many-valued function $y = \pm \arccos x + 2n\pi$, $n \in \mathbb{Z}$, which is denoted $y = \text{Arccos } x$.

Now we restrict the domain of tangent function to the open interval $\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$. The graph of this function is the branch of tangent function that passes through the origin (Figure 1.10). For this function with each value of $y \in (-\infty; \infty)$ is related exactly one value of x on the open interval $\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$.

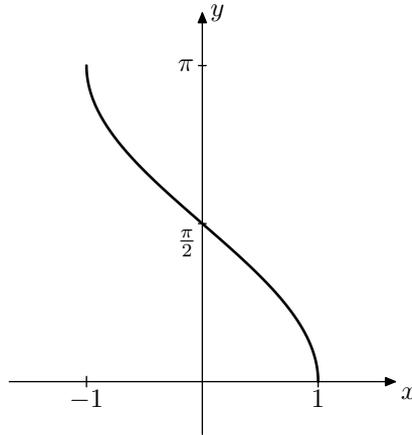


Figure 1.21: the function $y = \arccos x$

This function is called *arc tangent function* and denoted $x = \arctan y$. Exchanging x and y , we have the inverse function of $y = \tan x$, $x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$

$$y = \arctan x$$

. The domain of this function is $X = (-\infty; \infty)$ and the range $\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$.

If we solve the equation $y = \tan x$ for x in terms of y , we have $x = \arctan y + n\pi$, $n \in \mathbb{Z}$. Exchanging the notation, we obtain the inverse function of tangent function $y = \tan x$, which is infinitely many-valued function $y = \arctan x + n\pi$, $n \in \mathbb{Z}$ and denoted $y = \text{Arctan } x$.

Let us add to the three trigonometric functions considered the cotangent function $y = \cot x$. The graph of this function is in Figure 1.24.

We restrict the domain of the cotangent function $y = \cot x$ to the open interval $(0; \pi)$. With this restriction to each value of $y \in (-\infty; \infty)$ there corresponds exactly one value of $x \in (0; \pi)$. This function is called *arc cotangent function* and denoted $x = \text{arccot } y$. Exchanging x and y , we have the inverse function of $y = \cot x$, $x \in (0; \pi)$

$$y = \text{arccot } x$$

. The domain of this function is $X = (-\infty; \infty)$ and the range $Y = (0; \pi)$.

For the functions $y = \arctan x$ and $y = \text{arccot } x$ there holds the equality

$$\arctan x + \text{arccot } x = \frac{\pi}{2}$$

$$\forall x \in (-\infty; \infty).$$

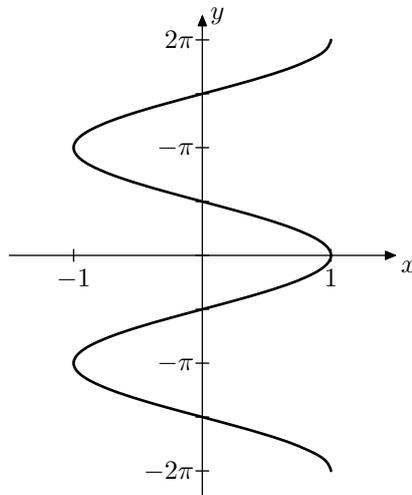


Figure 1.22: The function $y = \text{Arccos } x$

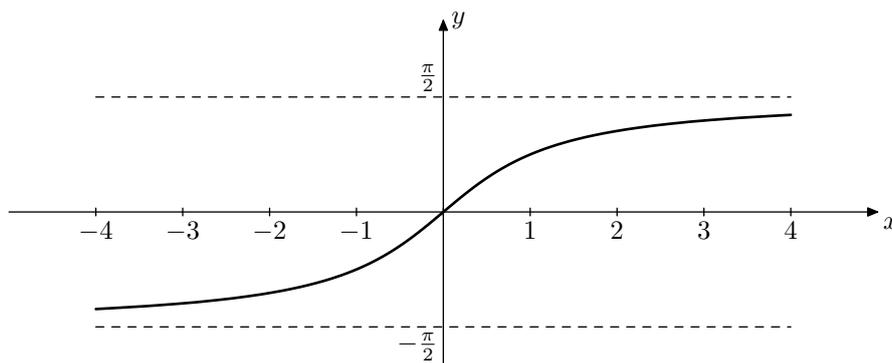


Figure 1.23: the function $y = \arctan x$

If we solve the equation $y = \cot x$ for x in terms of y , we obtain $x = \text{arccot } y + n\pi$, $n \in \mathbb{Z}$. Exchanging the notation we conclude that the inverse function of the cotangent function $y = \cot x$ is $y = \text{arccot } x + n\pi$, $n \in \mathbb{Z}$, denoted $y = \text{Arccot } x$.

Dealing with any of the functions considered, we can always see, that there holds the following statement.

The domain of the inverse is the range of the original function, and the range of the inverse is the domain of the original function.

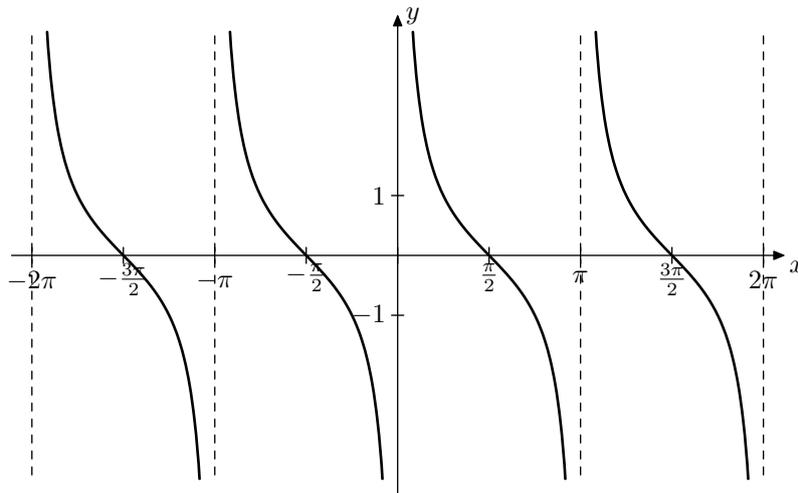


Figure 1.24: the function $y = \cot x$

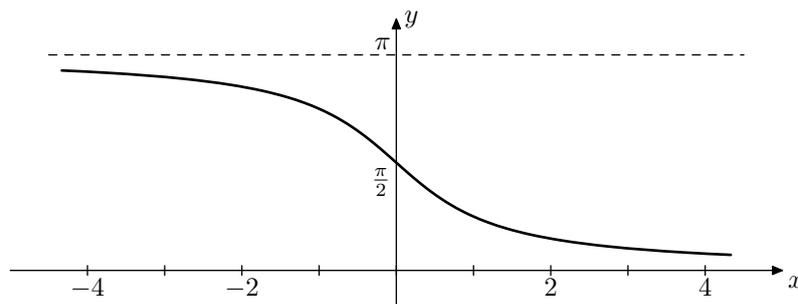


Figure 1.25: the function $y = \operatorname{arccot} x$

1.1.5 Hyperbolic functions and their inverse functions

In Calculus I we deal with some more elementary functions, called *hyperbolic functions* and with inverse hyperbolic functions. Hyperbolic functions are not the basic elementary functions as power functions, exponential functions, logarithmic functions etc. They are defined in terms of the exponential function.

The hyperbolic functions have similar names and similar properties to the trigonometric functions. In this subsection we define the four main hyperbolic functions, and sketch their graphs.

The hyperbolic functions are *hyperbolic sine*, *hyperbolic cosine*, *hyperbolic tangent* and *hyperbolic cotangent*.

The hyperbolic sine function $y = \sinh x$ is defined as

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

The graph of the hyperbolic sine function is sketched in Figure 1.26. The domain of the hyperbolic sine function is $X = (-\infty; \infty)$ and the range $Y = (-\infty; \infty)$.

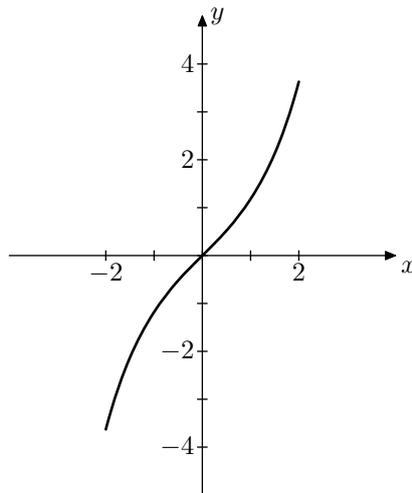


Figure 1.26: the hyperbolic sine function $y = \sinh x$

The hyperbolic sine is an odd function because $\forall x \in (-\infty; \infty)$

$$\sinh(-x) = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} = -\sinh x$$

and therefore is natural that the graph is symmetric with respect to origin.

The hyperbolic cosine function $y = \cosh x$ is defined as

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

The graph of the hyperbolic cosine is sketched in Figure 1.27. The domain of this function is $X = (-\infty; \infty)$ and the range $Y = [1; \infty)$.

The hyperbolic cosine is an even function because $\forall x \in (-\infty; \infty)$

$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh x$$

and the graph is symmetric with respect to y -axis.

Hyperbolic functions have identities which are similar to, but not the same as, the identities for trigonometric functions.

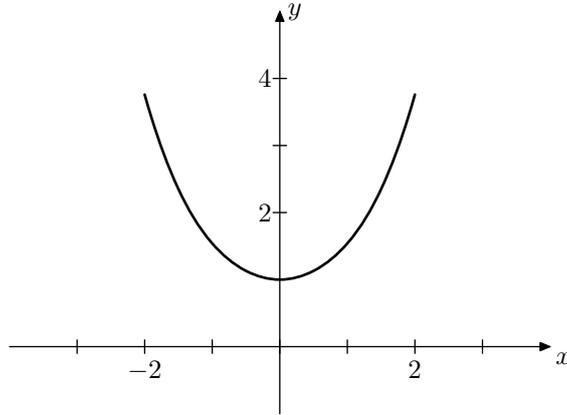


Figure 1.27: the hyperbolic cosine function $y = \cosh x$

1. $\cosh^2 x - \sinh^2 x = 1$;
2. $\sinh 2x = 2 \sinh x \cosh x$;
3. $\cosh 2x = \cosh^2 x + \sinh^2 x$.

Let us prove the first identity. For each $x \in (-\infty; \infty)$ we have

$$\begin{aligned}
 \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 = \\
 &= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} = \\
 &= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = \frac{4}{4} = 1
 \end{aligned}$$

which is what we wanted to prove.

Due to the first identity just proved, the hyperbolic functions $x = \sinh t$, $y = \cosh t$, $t \in (-\infty; \infty)$ parametrize the equilateral hyperbola $x^2 - y^2 = 1$.

The hyperbolic tangent $y = \tanh x$ is defined as

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

The graph of the hyperbolic tangent is in Figure 1.28. The domain of this function is $X = (-\infty; \infty)$ and the range $Y = (-1; 1)$.

The hyperbolic cotangent $y = \coth x$ is defined as

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

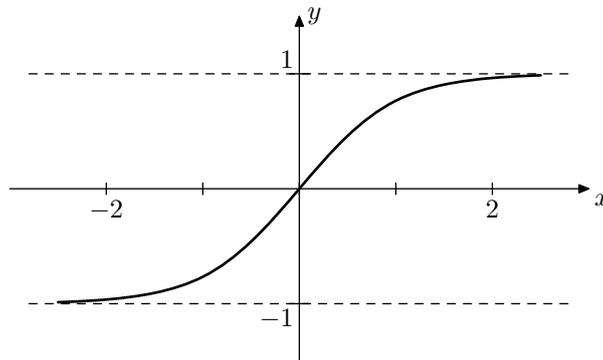


Figure 1.28: the hyperbolic tangent $y = \tanh x$

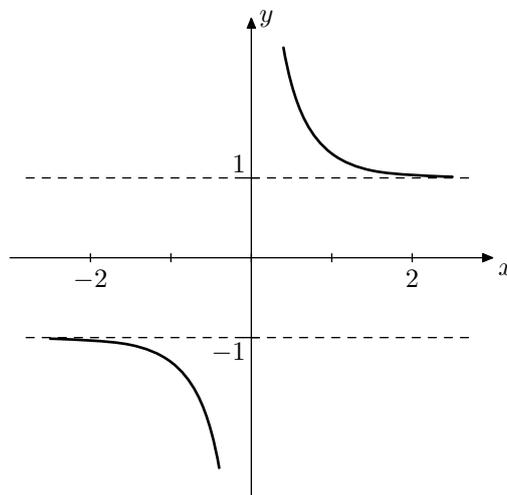


Figure 1.29: the hyperbolic cotangent $y = \coth x$

The graph of the hyperbolic cotangent is in Figure 1.29. The domain of this function is $X = (-\infty; 0) \cup (0; \infty)$ and the range $Y = (-\infty; -1) \cup (1; \infty)$.

To find the inverse function of $y = \sinh x$ we have to solve the equation $y = \frac{e^x - e^{-x}}{2}$ for x . Multiplying both sides of this equation by $2e^x$, we obtain $e^{2x} - 2ye^x - 1 = 0$, i.e. the quadratic equation with respect to e^x . The solution of this quadratic equation is

$$e^x = y \pm \sqrt{y^2 + 1}$$

The sign minus before the radical in this solution is impossible because $y - \sqrt{y^2 + 1} < 0$ but e^x cannot be negative. The only possibility is $e^x =$

$y + \sqrt{y^2 + 1}$ or

$$x = \ln(y + \sqrt{y^2 + 1})$$

If we exchange the notation, we obtain that the inverse function of the hyperbolic sine

$$y = \sinh x$$

is

$$y = \ln(x + \sqrt{x^2 + 1})$$

This function is called the *area sine function* and denoted $y = \operatorname{arsinh} x$. Thus,

$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$$

The domain of area sine function is the range of the hyperbolic sine, i.e. $X = (-\infty; \infty)$ and the range of the area sine function is the domain of hyperbolic sine function $Y = (-\infty; \infty)$.

The inverse function of hyperbolic cosine is clearly not uniquely determined. If we solve the equation $y = \frac{e^x + e^{-x}}{2}$ for x in terms of y , we obtain $x = \ln(y \pm \sqrt{y^2 - 1})$. Here we choose uniquely determined branch $x = \ln(y + \sqrt{y^2 - 1})$. Exchanging the notation, we have the inverse function of the hyperbolic cosine

$$y = \ln(x + \sqrt{x^2 - 1})$$

which is called *area cosine function* and denoted $y = \operatorname{arcosh} x$. By definition

$$\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1})$$

The domain of area cosine function is the range of hyperbolic cosine $X = [1; \infty)$. The range of the area cosine function is $Y = [0; \infty)$.

The expression of hyperbolic tangent function is $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. If we solve this equation for x in terms of y , we have $x = \frac{1}{2} \ln \frac{1+y}{1-y}$. Exchanging the variables we obtain that the inverse function of the hyperbolic tangent $y = \tanh x$ is *area tangent function* $y = \frac{1}{2} \ln \frac{1+x}{1-x}$ denoted

$$y = \operatorname{artanh} x$$

The domain of area tangent function is $X = (-1; 1)$ and the range $Y = (-\infty; \infty)$ (see the domain and the range of the hyperbolic tangent function).

1.1.6 Composite function

Let us suppose that the function $u = g(x)$ associates with each value $x \in X$ one certain value of $u \in U$. Here the dependent variable $u \in U$ can serve as an independent variable for another function $y = f(u)$ that relates each element of U with one certain value of y .

If we substitute u in terms of x , we obtain the *composite function*

$$y = f[g(x)]$$

Sometimes the composite function is denoted $f \circ g$ that means

$$(f \circ g)(x) = f[g(x)]$$

Here $u = g(x)$ is called *internal function* and $y = f(u)$ *external function*. The composite function can be useful if the original function is too complicated to work on.

Example 1. The internal function of the composite function $y = \sqrt{1 - x^2}$ is

$$u = 1 - x^2$$

and the external function

$$y = \sqrt{u}$$

Example 2. Let us compose the composite functions $f[g(x)]$, $g[f(x)]$, $f[f(x)]$ and $g[g(x)]$ if $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$.

The first

$$f[g(x)] = [g(x)]^2 + 1 = (\sqrt{x})^2 + 1 = x + 1$$

the second

$$g[f(x)] = \sqrt{f(x)} = \sqrt{x^2 + 1}$$

the third

$$f[f(x)] = [f(x)]^2 + 1 = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 1 + 1 = x^4 + 2x^2 + 2$$

and the last

$$g[g(x)] = \sqrt{g(x)} = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

Example 3. For given internal function $u = \frac{1+x}{1-x}$ and external function $y = \frac{1}{2} \ln u$ the composite function

$$y = \frac{1}{2} \ln \frac{1+x}{1-x}$$

i.e.

$$y = \operatorname{artanh} x.$$

We can form the composite functions of more than two functions. If u is a function of x

$$u = h(x)$$

the variable v is a function of u

$$v = g(u)$$

and the variable y is a function of v ,

$$t = f(v)$$

we can build the composite function

$$y = f\{g[h(x)]\}$$

Analogously we can build the composite functions of four, five etc. functions.

The domain of the composite function consists of those values of the independent variable for which the result yielded by the internal function lies in the domain of the external function.

Example 4. Let us form the composite function of $u = \cos x$, $v = \log u$ and $y = \sqrt{v}$ and find the domain of this function.

Substituting u into second function, we have $v = \log \cos x$ and substituting this result into third function, we obtain

$$y = \sqrt{\log \cos x}$$

To find the domain of this function, we have the restriction

$$\log \cos x \geq 0$$

or

$$\cos x \geq 1$$

The last inequality holds only if $\cos x = 1$, i.e.

$$x = 0, \pm 2\pi, \pm 4\pi, \dots$$

that means the domain of the composite function consists of the separated points $X = \{x | x = 2n\pi, n \in \mathbb{Z}\}$