5 Definite integral

5.1 Definition of definite integral

There are many ways of formally defining an integral, not all of which are equivalent. The differences exist mostly to deal with differing special cases which may not be integrable under other definitions, but also occasionally for pedagogical reasons. The most commonly used definition of integral is Riemann integral.

Assume that the function $y = f(x)$ is defined on the closed interval $[a; b]$. Let

$$a = x_0 < x_1 < x_2 < \ldots < x_{k-1} < x_k < \ldots < x_n = b$$

be an arbitrary (randomly selected) partition of the interval $[a; b]$, which divides the interval into n subintervals

$$[x_{k-1}; x_k], \text{ where } k = 1,\ 2,\ \ldots,\ n$$

Denote by $\Delta x_k = x_k - x_{k-1}$ the length of the kth subinterval.

Further we choose on any of these subintervals an arbitrary point

$$\xi_k \in [x_{k-1}; x_k],\ k = 1,\ 2,\ \ldots,\ n$$

and multiply the value of the function at ξ_k by the length of the kth subinterval:

$$f(\xi_k)\Delta x_k,\ k = 1,\ 2,\ \ldots,\ n.$$

Assuming $f(x) \geq 0$ on $[a; b]$, this product is the area of the rectangle with height equal to the function value at the distinguished point ξ_k of the kth sub-interval, and width the same as the sub-interval width Δx_k.

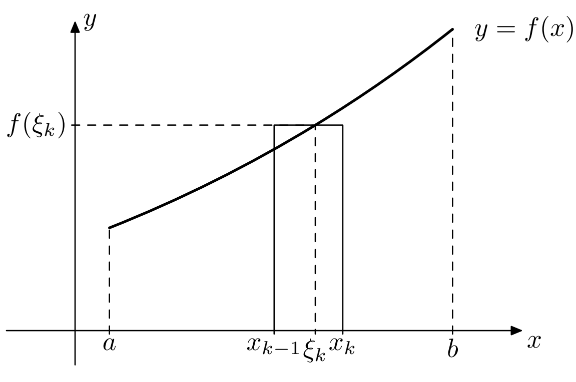


Figure 5.1: the area of the rectangle $f(\xi_k)\Delta x_k$

If we add these products, we obtain the sum

$$s_n = \sum_{k=1}^{n} f(\xi_k)\Delta x_k$$

which is called the *integral sum* of the function $f(x)$ over the interval $[a; b]$.

The dividing points $x_1,\ x_2,\ \ldots$ are chosen arbitrarily. Hence, the subintervals have the different length $\Delta x_k,\ k = 1,\ 2,\ \ldots,\ n$. Let λ denote the greatest length of these subintervals

$$\lambda = \max_{1 \le k \le n} \Delta x_k$$

Definition 1. If the limit

$$\lim_{\lambda \to 0} s_n$$

exists and does not depend on the partition of the interval $[a; b]$ and does not depend on the choice of the points $\xi_k \in [x_{k-1}; x_k]$, then this limit is called the *definite integral* of the function $f(x)$ from a to b and denoted

$$\int_a^b f(x)dx$$

The symbol $\int$ is the integral sign, a is called *lower limit*, b is called *upper limit*, $f(x)$ is called *integrand* and dx is called the *differential* of the argument, x is called the *variable of integration*.

Definition 2. If the conditions of the definition 1 are satisfied then $f(x)$ is called the *integrable function* on $[a; b]$.

There holds the following theorem.

Theorem 1. If the function $f(x)$ is continuous on $[a; b]$ then it is integrable on $[a; b]$.

Remark. If the function is discontinuous on $[a; b]$ then it may be integrable or may be not integrable on $[a; b]$.

By definition

$$\int_a^b f(x)dx = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k$$

If $f(x) \ge 0$ on the interval $[a; b]$, then the products $f(\xi_k)\Delta x_k$ in the integral sum are the ares of the rectangles with height $f(\xi_k)$ and width Δx_k. Hence, the integral sum represents approximately the area under the graph of the function $y = f(x)$, i.e. area of the domain in xy-plane bounded by

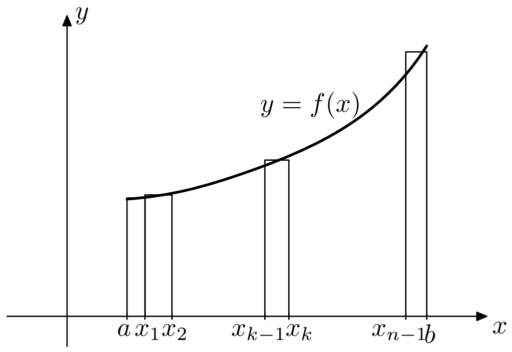


Figure 5.2: The integral sum

the graph of the function $y = f(x)$, x-axis and two vertical lines $x = a$ and $x = b$.

If $\lambda \to 0$, then the length of any subinterval gets shorter and, to cover the given interval $[a; b]$, the number of those subintervals has to increase. The integral sum - the sum of the areas of the rectangles - is getting closer to the area under the graph of the function $y = f(x)$.

Therefore, if $f(x) \geq 0$ on the interval $[a; b]$ then *the definite integral is the area under the graph of the function* $y = f(x)$.

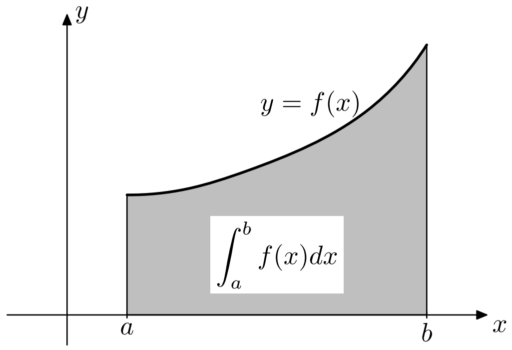


Figure 5.3: The area under the graph of the function $y = f(x)$

5.2 Properties of definite integral

Property 1 The definite integral of the sum of two functions is equal to the sum of the definite integrals of these functions:

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx. \tag{2.1}$$

Proof By the definition of the definite integral

$$I = \int_a^b [f(x) + g(x)]dx = \lim_{\lambda \to 0} \sum_{k=1}^{n} [f(\xi_k) + g(\xi_k)]\Delta x_k$$

Removing the square brackets under the sum and taking into account that the sum does not depend on the order of summation, we obtain

$$I = \lim_{\lambda \to 0} \left[\sum_{k=1}^{n} f(\xi_k)\Delta x_k + \sum_{k=1}^{n} g(\xi_k)\Delta x_k \right]$$

The limit of the sum equals to the sum of the limits, i.e.

$$I = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k + \lim_{\lambda \to 0} \sum_{k=1}^{n} g(\xi_k)\Delta x_k$$

By the definition of the definite integral these limits are the definite integrals on the right side of (2.1).

Property 2 The constant coefficient c can be factored out:

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

The proof is similar to the proof of the first property.

Conclusion 1. The definite integral of the difference of two functions equals to the difference of the definite integrals of those functions:

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

Proof It follows from the first and the second property. Writing $f(x) - g(x) = f(x) + (-1)g(x)$ gives

$$\int_a^b [f(x) - g(x)]dx = \int_a^b [f(x) + (-1)g(x)]dx = \int_a^b f(x)dx + (-1)\int_a^b g(x)dx$$

which is we wanted to prove.

Property 3 If $f(x) \geq 0$ for any $x \in [a; b]$, then

$$\int_a^b f(x)dx \geq 0$$

Proof. If $f(x) \geq 0$ on $[a; b]$, then $f(x) \geq 0$ on any subinterval $[x_{k-1}; x_k]$, $k = 1,\ 2,\ \ldots,\ n$. Thus, for $\xi_k \in [x_{k-1}; x_k]$ also $f(\xi_k) \geq 0$. Multiplying the last inequality by the length of the kth subinterval gives $f(\xi_k)\Delta x_k \geq 0$, $k = 1,\ 2,\ \ldots,\ n$.

Adding n nonnegative quantities, we obtain nonnegative quantity

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k \geq 0$$

By the limit theorem the limit of the nonnegative quantity as $\lambda \to 0$ is nonnegative, which proves the property.

Conclusion 2. If $f(x) \leq g(x)$ for any $x \in [a; b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Proof. By assumption $g(x) - f(x) \geq 0$. Hence, by the Property 3

$$\int_a^b [g(x) - f(x)]dx \geq 0$$

By Conclusion 1

$$\int_a^b g(x)dx - \int_a^b f(x)dx \geq 0$$

which proves the statement.

Property 4. The absolute value of the definite integral of the function $f(x)$ is less than, or equal to, the definite integral of the absolute value of this function:

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

Proof. Here we use the property of the absolute value of the sum $|a+b| \leq |a| + |b|$ for n addends. By the definition of the definite integral

$$\left|\int_a^b f(x)dx\right| = \left|\lim_{\lambda\to 0}\sum_{k=1}^{n} f(\xi_k)\Delta x_k\right| = \lim_{\lambda\to 0}\left|\sum_{k=1}^{n} f(\xi_k)\Delta x_k\right| \leq$$

$$\leq \lim_{\lambda\to 0}\sum_{k=1}^{n} |f(\xi_k)\Delta x_k| = \lim_{\lambda\to 0}\sum_{k=1}^{n} |f(\xi_k)|\Delta x_k = \int_a^b |f(x)|dx$$

Property 5. If we change the limits of integration, then the sign of the integral changes:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Proof. If we define the definite integral $\int_b^a f(x)dx$, then the start point is b. If we assume that the definite integrals in this property exist, the limit does not depend on the partition. So in both definitions we can use the same partition. As well we can use in both definitions the same arbitrarily chosen points $\xi_1,\ \xi_2,\ \ldots,\ \xi_{k-1},\ \xi_k,\ \ldots, \xi_n,$ because the limit does not depend on the choice of these points. If we move in direction from b to a, then the first partition point is $x_n = b$, next x_{n-1}, ..., x_k, x_{k-1}, ..., $x_0 = a$. The start point of the kth subinterval x_k and the endpoint x_{k-1}. By the definition of the definite integral

$$\int_b^a f(x)dx = \lim_{\lambda\to 0}\sum_{k=n}^{1} f(\xi_k)(x_{k-1} - x_k)$$

The integral sum in this definition is

$$\sum_{k=n}^{1} f(\xi_k)(x_{k-1} - x_k) = \sum_{k=n}^{1} f(\xi_k)(-\Delta x_k) = -\sum_{k=n}^{1} f(\xi_k)\Delta x_k$$

and, because the sum does not depend on the order of addition,

$$= \sum_{k=n}^{1} f(\xi_k)(x_{k-1} - x_k) = -\sum_{k=1}^{n} f(\xi_k)\Delta x_k$$

The limit of the left side of this equality as $\lambda \to 0$ is $\int\limits_b^a f(x)dx$ and the limit of the right side is $-\int\limits_a^b f(x)dx$.

Conclusion 3. If the lower and upper limit of the definite integral are equal, then the definite integral equals to zero:

$$\int\limits_a^a f(x)dx = 0$$

Proof. Changing the limits of integration, we have by Property 5

$$\int\limits_a^a f(x)dx = -\int\limits_a^a f(x)dx$$

or

$$2\int\limits_a^a f(x)dx = 0$$

which yields the assertion.

Property 6 (Additivity property of the definite integral).

$$\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx$$

Proof. First we assume that c is in the interval $[a;b]$, i.e. $a < c < b$. Defining the integral on the left side of this equality, we choose an arbitrary partition of the interval $[a;b]$, so that the first partition point is c. The further arbitrary partition of $[a;b]$ produces an arbitrary partition of the intervals $[a;c]$ and $[c;b]$. Thus, the integral sum for the whole interval $[a;b]$ can be written as the sum of the two integral sums

$$\sum_{[a;b]} f(\xi_k)\Delta x_k = \sum_{[a;c]} f(\xi_k)\Delta x_k + \sum_{[c;b]} f(\xi_k)\Delta x_k$$

If the greatest length of the subintervals of $[a;b]$ $\lambda \to 0$, then the greatest lengths of the subintervals of $[a;c]$ and $[c;b]$ approach zero as well. Therefore, taking the limits as $\lambda \to 0$ on both sides completes the proof.

If c is outside the interval $[a;b]$, suppose $c > b > a$, then

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

It follows

$$\int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx$$

and changing the limits we have by Property 5

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

In similar way we can prove that this property holds if $c < a$.

Property 7. If m is the least value of $f(x)$ and M is the greatest value of $f(x)$ on the interval $[a;b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Proof. The proofs of these two inequalities are similar and we prove only the right hand inequality.

As assumed, the greatest value of the function $f(x)$ on $[a;b]$ is M. Thus, $f(\xi_k) \leq M$ for any arbitrarily chosen $\xi_k \in [x_{k-1}; x_k]$ for each $k = 1, 2, \ldots, n$. Multiplying this inequality by Δx_k gives

$$f(\xi_k)\Delta x_k \leq M\Delta x_k$$

Adding these products, we obtain

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k \leq \sum_{k=1}^{n} M\Delta x_k =$$
$$= M(x_1 - x_0 + x_2 - x_1 + x_3 - x_2 + \ \ldots \ + x_n - x_{n-1}) = M(b-a),$$

because $x_0 = a$ and $x_n = b$.

There is constant on the right side of the inequality

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k \leq M(b-a)$$

and taking the limit on both sides of this inequality as $\lambda \to 0$ gives the assertion.

Property 8 (Mean value property of the definite integral). If the function $f(x)$ is continuous on $[a; b]$, then there exists at least one point $\xi \in [a; b]$ such that

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

Proof. The function continuous in the closed interval has the least value m and the greatest value M on this interval, hence, there holds the Property 7. Dividing the both sides of the both inequalities by the length of the interval of integration $b-a$ gives

$$m \leq \frac{1}{b-a} \int_a^b f(x)dx \leq M$$

Consequently,

$$\frac{1}{b-a} \int_a^b f(x)dx$$

is between the least and the greatest value. The function continuous on $[a; b]$ has any value between the least and the greatest. Therefore, there exists at least one point $\xi \in [a; b]$, where the function obtains this value, that is

$$f(\xi) = \frac{1}{b-a} \int_a^b f(x)dx. \tag{2.2}$$

The multiplication of both sides of this equality by $b-a$ completes the proof.

The value $f(\xi)$ is called the *mean value* of the function $f(x)$ on the interval $[a; b]$. This value is computed by (2.2).

5.3 Computation of definite integral. Newton-Leibnitz formula

Suppose $f(x)$ is defined on $[a; b]$. Let us define on $[a; b]$ the function of the upper limit of the definite integral

$$\Phi(x) = \int_a^x f(t)dt \tag{3.3}$$

Theorem 1. If the function $f(x)$ is continuous on $[a;b]$, then $\Phi(x)$ is differentiable on $(a;b)$ and $\Phi'(x) = f(x)$.

Proof. We use the definition of the derivative of $\Phi(x)$

$$\Phi'(x) = \lim_{\Delta x \to 0} \frac{\Phi(x + \Delta x) - \Phi(x)}{\Delta x}$$

By additivity property of the definite integral,

$$\Phi(x + \Delta x) - \Phi(x) = \int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt =$$

$$\int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt - \int_a^x f(t)dt = \int_x^{x+\Delta x} f(t)dt.$$

As assumed, the function $f(x)$ is continuous on $[a;b]$. Hence, by the mean value property, there exists $\xi \in [x; x + \Delta x]$ such that

$$\Phi(x + \Delta x) - \Phi(x) = f(\xi)(x + \Delta x - x) = f(\xi)\Delta x$$

Consequently

$$\frac{\Phi(x + \Delta x) - \Phi(x}{\Delta x} = f(\xi)$$

In the definition of the derivative $\Delta x \to 0$. It follows that $x + \Delta x \to x$ and since ξ is a point between x and $x + \Delta x$, then $\xi \to x$ also. Thus,

$$\Phi'(x) = \lim_{\Delta x \to 0} f(\xi) = \lim_{\xi \to x} f(\xi)$$

and the third condition of continuity of $f(x)$ gives $\Phi'(x) = f(x)$, which is we wanted to prove.

Remark. In some textbooks the Theorem 1 is referred as the *Fundamental Theorem of Calculus.*

By Theorem 1, the function $\Phi(x)$ is an antiderivative of $f(x)$. If $F(x)$ is the known antiderivative of $f(x)$ (by the table of integrals or by some technique of integration), then by Corollary 1.2 of 4.1 the antiderivatives $\Phi(x)$ and $F(x)$ differ at most by a constant, i.e. $\Phi(x) = F(x)+C$. According to the definition of $\Phi(x)$

$$F(x) + C = \int_a^x f(t)dt. \tag{3.4}$$

Taking in this equality $x = a$, we obtain by Conclusion 3 of previous subsection

$$F(a) + C = \int_a^a f(t)dt = 0$$

which yields $C = -F(a)$. Substituting C to (3.4) gives

$$F(x) - F(a) = \int_a^x f(t)dt$$

and taking in the last equality $x = b$, we obtain

$$F(b) - F(a) = \int_a^b f(t)dt. \tag{3.5}$$

Consequently, the antiderivative familiar from the indefinite integral is the appropriate tool to evaluate the definite integral. Now we take in (3.5) for the variable of integration x again. To facilitate the computation we use the notation

$$F(b) - F(a) = F(x)\Big|_a^b$$

Finally, we formulate the result obtained as a theorem.

Theorem 2. If the function $f(x)$ is continuous on $[a; b]$ and $F(x)$ is the antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a), \tag{3.6}$$

The formula (3.6) is called *Newton-Leibnitz formula.*

Example 1. Evaluate

$$\int_1^e \frac{dx}{x} = \ln x\Big|_1^e = \ln e - \ln 1 = 1$$

Example 2. Evaluate

$$\int_0^1 \frac{xdx}{\sqrt{1+x^2}}$$

For the integration we use the equality $d(1+x^2) = 2xdx$ and find

$$\int_0^1 \frac{xdx}{\sqrt{1+x^2}} = \frac{1}{2}\int_0^1 \frac{2xdx}{\sqrt{1+x^2}} = \frac{1}{2}\int_0^1 \frac{d(1+x^2)}{\sqrt{1+x^2}} =$$

$$= \frac{1}{2} \cdot 2\sqrt{1+x^2}\Big|_0^1 = \sqrt{2} - 1.$$

Example 3. Compute the mean value of the function $f(x) = x^2$ on $[1; 3]$. By the mean value formula (2.2), we find

$$\frac{1}{3-1}\int_1^3 x^2dx = \frac{1}{2}\frac{x^3}{3}\Big|_1^3 = \frac{1}{2}\left(\frac{27}{3} - \frac{1}{3}\right) = \frac{13}{3} = 4\frac{1}{3}$$

5.4 Change of variable in definite integral

The choice of the new variable depends on the function to be integrated. These principals are familiar from the indefinite integral.

If we compute the definite integral, we are interested in its value, not in the antiderivative of the initial function. This is because after the integration by change of variable in the definite integral we don't re-substitute the initial variable. Instead of it we compute the limits of integration for the new variable.

Changing the variable $x = \varphi(t)$ in the definite integral

$$\int_a^b f(x)dx$$

we find $dx = \varphi'(t)dt$. The equation $\varphi(t) = a$ gives the lower limit for the new variable $t = \alpha$ and the equation $\varphi(t) = b$ gives the upper limit $t = \beta$. The change of variable formula is

$$\int_a^b f(x)dx = \int_\alpha^\beta f[(\varphi(t)]\varphi'(t)dt$$

Example 4. Compute $I = \int_0^2 \sqrt{8-x^2}dx$. To remove the irrationality we change the variable $x = 2\sqrt{2}\sin t$. Then $dx = 2\sqrt{2}\cos tdt$ and

$$\sqrt{8-x^2} = \sqrt{8 - 8\sin^2 t} = \sqrt{8\cos^2 t} = 2\sqrt{2}\cos t$$

We determine the limits for the new variable t. If $x = 0$, then $\sin t = 0$, it follows $t = 0$. If $x = 2$, then $2\sqrt{2}\sin t = 2$ or $\sin t = \dfrac{\sqrt{2}}{2}$, hence, $t = \dfrac{\pi}{4}$. Thus,

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} 2\sqrt{2}\cos t \cdot 2\sqrt{2}\cos t dt = 8\int_0^{\frac{\pi}{4}} \cos^2 t dt = \\ &= 4\int_0^{\frac{\pi}{4}} (1+\cos 2t)dt = 4\int_0^{\frac{\pi}{4}} dt + 2\int_0^{\frac{\pi}{4}} \cos 2t d(2t) = \\ &= 4t\Big|_0^{\frac{\pi}{4}} + 2\sin 2t\Big|_0^{\frac{\pi}{4}} = \pi + 2. \end{aligned}$$

5.5 Integration by parts

Let $u(x)$ and $v(x)$ be two differentiable function on $[a;b]$. The differential of the product of these functions

$$d(uv) = [u(x)v(x)]' = [u'(x)v(x)+u(x)v'(x)]dx = u'(x)v(x)dx+u(x)v'(x)dx = udv+vdu$$

By Conclusion 1.6 of subsection 4.1 the product uv is one of the antiderivatives of $d(uv)$. Integration of the equality $d(uv) = udv + vdu$ over $[a;b]$ gives

$$uv\Big|_a^b = \int_a^b udv + \int_a^b vdu$$

which yields

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu \tag{5.7}$$

This is the formula of the integration by parts for the definite integral. The choice of the function u and the differential dv is the same as in case of the indefinite integral.

Example 5. Compute $\displaystyle\int_1^e \ln x dx$

Here we choose $u = \ln x$ and $dv = dx$. Hence, $du = \dfrac{dx}{x}$ and $v = x$. By the integration by parts formula (5.7),

$$\int\limits_1^e \ln x dx = x\ln x\Big|_1^e - \int\limits_1^e x\cdot\frac{dx}{x} = e - x\Big|_1^e = e-(e-1)=1$$

5.6 Improper integral over infinite interval

In this section we consider a couple of different kinds of integrals. Both of these are integrals that are called improper integrals. In the first kind of improper integrals one or both of the limits of integration are infinity.

Definition 1. Let the function $f(x)$ be defined and continuous on the infinite interval $[a;\infty)$. If for any $N\in[a;\infty)$ there exists the definite integral $\int\limits_a^N f(x)dx$ and there exists the limit $\lim\limits_{N\to\infty}\int\limits_a^N f(x)dx$, then this limit is called the improper integral with the infinite upper limit and denoted $\int\limits_a^\infty f(x)dx$.

By Definition 1

$$\int\limits_a^\infty f(x)dx = \lim_{N\to\infty}\int\limits_a^N f(x)dx \tag{6.8}$$

If the limit exists and is a finite number, then the improper integral is said to be *convergent*. If the limit does not exist or the limit is infinite, then the improper integral is said to be *divergent*.

Thus, to compute the improper integral, we first have to compute the definite integral over $[a;N]$ and next find the limit of this result as $N\to\infty$.

Example 6. Evaluate $\int\limits_0^\infty \frac{dx}{1+x^2}$.

By the formula (6.8),

$$\int\limits_0^\infty \frac{dx}{1+x^2} = \lim_{N\to\infty}\int\limits_0^N \frac{dx}{1+x^2} = \lim_{N\to\infty}\arctan x\Big|_0^N = \lim_{N\to\infty}(\arctan N - \arctan 0) = \frac{\pi}{2}$$

So, this improper integral is convergent.

Definition 2. Let the function $f(x)$ be defined and continuous on the infinite integral $(-\infty;b]$. If for any $M\in(-\infty;b]$ there exists $\int\limits_M^b f(x)dx$ and there exists the limit $\lim\limits_{M\to-\infty}\int\limits_{M}^{b} f(x)dx$, then this limit is called the improper integral with the infinite lower limit and denoted $\int\limits_{-\infty}^{b} f(x)dx$.

By Definition 2,

$$\int\limits_{-\infty}^{b} f(x)dx = \lim_{M\to-\infty}\int\limits_{M}^{b} f(x)dx \tag{6.9}$$

The convergence and divergence of this improper integral are defined in the same way as in the pervious case.

Definition 3. If the function $f(x)$ is defined and continuous in $(-\infty;\infty)$, then the improper integral over $(-\infty;\infty)$ is defined as

$$\int\limits_{-\infty}^{\infty} f(x)dx = \int\limits_{-\infty}^{c} f(x)dx + \int\limits_{c}^{\infty} f(x)dx$$

where c is any finite real number.

If both of the improper integrals on the right side of this equality are convergent, then this improper integral is said to be convergent. If at least one of the improper integrals on the right side of this equality is divergent, then this improper integral is said to be divergent.

Example 7. Let $a > 0$ and let us decide for which values of α the improper integral $\int\limits_{a}^{\infty}\frac{dx}{x^\alpha}$ is convergent and for which values of α it is divergent.

Denote this improper integral by I and find

$$I = \int\limits_{a}^{\infty}\frac{dx}{x^\alpha} = \lim_{N\to\infty}\int\limits_{a}^{N}\frac{dx}{x^\alpha}$$

If $\alpha \neq 1$, then

$$I = \lim_{N\to\infty}\frac{x^{-\alpha+1}}{-\alpha+1}\bigg|_{a}^{N} = \lim_{N\to\infty}\left(\frac{N^{-\alpha+1}}{-\alpha+1} - \frac{a^{-\alpha+1}}{-\alpha+1}\right)$$

If $\alpha > 1$, then

$$\lim_{N\to\infty}\left(\frac{1}{(1-\alpha)N^{\alpha-1}} - \frac{1}{(1-\alpha)a^{\alpha-1}}\right) = \frac{1}{(\alpha-1)a^{\alpha-1}}$$

that means, the improper integral is convergent. If $\alpha < 1$, then

$$\lim_{N\to\infty}\left(\frac{N^{1-\alpha}}{(1-\alpha)}-\frac{a^{1-\alpha}}{1-\alpha}\right)=\infty$$

i.e. the improper integral is divergent.

If $\alpha = 1$, then

$$\int_a^{\infty}\frac{dx}{x}=\lim_{N\to\infty}\ln x\Big|_a^N=\lim_{N\to\infty}(\ln N-\ln a)=\infty$$

thus, the improper integral is divergent again.

Consequently, the improper integral $\int\limits_a^{\infty}\frac{dx}{x^{\alpha}}$ is convergent, if $\alpha > 1$ and divergent, if $\alpha \leq 1$.

In many cases we are rather interested in the convergence of the improper integral than in the actual value of this integral. Moreover, sometimes an improper integral is too difficult to evaluate, but we still need to know, is it convergent or not. One technique is to compare it with a known integral. The theorems below, called the *comparison theorems*, enable us to decide whether the improper integral is convergent or divergent. We formulate these theorems for the improper integral with infinite upper limit. These theorems hold as well for the improper integrals with infinite lower limit and in case, if both limits are infinite.

We assume, that we know whether the improper integral $\int\limits_a^{\infty}\varphi(x)dx$ is convergent or divergent.

Theorem 3. Suppose that $f(x)$ and $\varphi(x)$ are two continuous on $[a;\infty)$ functions such that $0 \leq f(x) \leq \varphi(x)$ on this interval. Then the convergence of the improper integral

$$\int_a^{\infty}\varphi(x)dx \tag{6.10}$$

yields the convergence of the improper integral

$$\int_a^{\infty}f(x)dx. \tag{6.11}$$

Suppose that $f(x)$ and $\varphi(x)$ are two continuous on $[a;\infty)$ functions such that $0 \leq \varphi(x) \leq f(x)$ on this interval. Then the divergence of the improper integral (6.10) yields the divergence of the improper integral (6.11).

Theorem 4. Suppose that two continuous on $[a; \infty)$ functions $f(x)$ and $\varphi(x)$ are equivalent in the limiting process $x \to \infty$. Then the convergence of (6.10) yields the convergence of (6.11) and the divergence of (6.10) yields the divergence of (6.11).

Definition 4. The improper integral (6.11) is called *absolutely convergent*, if the improper integral

$$\int_a^{\infty} |f(x)| dx$$

is convergent.

Theorem 5. The absolute convergence of (6.11) yields the convergence of this improper integral.

Example 8. Decide on the convergence or divergence of

$$\int_1^{\infty} \frac{\arctan x dx}{1+x^2}$$

In the half-interval $[0; \infty)$ there holds $\arctan x \leq \dfrac{\pi}{2}$. By Example 6, the improper integral $\displaystyle\int_0^{\infty} \frac{dx}{1+x^2}$ is convergent. Applying Theorem 3 for $f(x) = \dfrac{\arctan x}{1+x^2}$ and $\varphi(x) = \dfrac{\pi}{2} \cdot \dfrac{1}{1+x^2}$, we conclude that the given improper integral is convergent.

Example 9. Decide on the convergence or divergence of $\displaystyle\int_2^{\infty} \frac{dx}{x-1}$.

In the limiting process $x \to \infty$, the functions $f(x) = \dfrac{1}{x-1}$ and $\varphi(x) = \dfrac{1}{x}$ are equivalent because

$$\lim_{x \to \infty} \frac{\frac{1}{x-1}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{x}{x-1} = 1$$

By Example 7 the improper integral $\displaystyle\int_2^{\infty} \frac{dx}{x}$ is divergent. Thus, by Theorem 4, the given improper integral is also divergent.

Example 10. Decide on the convergence or divergence of $\displaystyle\int_1^{\infty} \frac{\sin x dx}{x^2}$.

For any $x \in \mathbb{R}$ there holds $\left|\dfrac{\sin x}{x^2}\right| \leq \dfrac{1}{x^2}$. By Example 7 the improper integral $\int\limits_2^{\infty} \dfrac{dx}{x^2}$ is convergent. By Theorem 3 this improper integral is absolutely convergent and by Theorem 5 it is convergent.

5.7 Improper integrals of unbounded functions

Suppose that the function $f(x)$ is unbounded in a neighborhood of the right endpoint b of the interval $[a; b]$.

Definition 5. If for any $\varepsilon > 0$ there exists the definite integral $\int\limits_a^{b-\varepsilon} f(x)dx$ and there exists the limit $\lim\limits_{\varepsilon \to 0} \int\limits_a^{b-\varepsilon} f(x)dx$, then this limit is called the improper integral of the unbounded function at the upper limit and denoted $\int\limits_a^b f(x)dx$.

By Definition 5 we evaluate the improper integral of the unbounded function in the neighborhood of the upper limit b, using the formula

$$\int\limits_a^b f(x)dx = \lim_{\varepsilon \to 0} \int\limits_a^{b-\varepsilon} f(x)dx \tag{7.12}$$

Improper integrals are often written symbolically just like standard definite integrals.

Suppose that the function $f(x)$ is unbounded in a neighborhood of the left endpoint a of the interval $[a; b]$.

Definition 6. If for any $\varepsilon > 0$ there exists the definite integral $\int\limits_{a+\varepsilon}^{b} f(x)dx$ and there exists the limit $\lim\limits_{\varepsilon \to 0} \int\limits_{a+\varepsilon}^{b} f(x)dx$, then this limit is called the improper integral of the unbounded function at the lower limit and denoted $\int\limits_a^b f(x)dx$.

By Definition 6 the improper integral of the unbounded function at the lower limit a we evaluate by the formula

$$\int\limits_a^b f(x)dx = \lim_{\varepsilon\to 0}\int\limits_{a+\varepsilon}^b f(x)dx \qquad (7.13)$$

If the function $f(x)$ is unbounded in some interior point c of $[a;b]$, then we use the additivity property on the integral and write

$$\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx$$

and evaluate the first addend by (7.12) and the second addend by (7.13)

If the limits in (7.12) and (7.13) are finite, then the improper integral is said to be convergent. If these limits either does not exist or are infinite, then this improper integral is said to be divergent.

Definition 7. The improper integral of the unbounded function is said to be absolutely convergent if the improper integral

$$\int\limits_a^b |f(x)|dx$$

is convergent.

Example 11. Let us find how the convergence or divergence of

$$\int\limits_a^b \frac{dx}{(b-x)^\alpha} \qquad (7.14)$$

depends on the exponent α.

The integrand $\dfrac{1}{(b-x)^\alpha}$ is unbounded in the neighborhood of the upper limit b. By formula (7.12)

$$\int\limits_a^b \frac{dx}{(b-x)^\alpha} = \lim_{\varepsilon\to 0}\int\limits_a^{b-\varepsilon} \frac{dx}{(b-x)^\alpha}.$$

Suppose $\alpha \neq 1$. Using the equality of the differentials $d(b-x) = -dx$, we obtain

$$\lim_{\varepsilon\to 0}\int\limits_a^{b-\varepsilon}\frac{dx}{(b-x)^\alpha} = -\lim_{\varepsilon\to 0}\int\limits_a^{b-\varepsilon}(b-x)^{-\alpha}d(b-x) = -\lim_{\varepsilon\to 0}\left.\frac{(b-x)^{-\alpha+1}}{-\alpha+1}\right|_a^{b-\varepsilon} =$$

$$= -\lim_{\varepsilon\to 0}\left[\frac{\varepsilon^{-\alpha+1}}{-\alpha+1}-\frac{(b-a)^{-\alpha+1}}{-\alpha+1}\right] = \lim_{\varepsilon\to 0}\left[\frac{(b-a)^{1-\alpha}}{1-\alpha}-\frac{\varepsilon^{1-\alpha}}{1-\alpha}\right].$$

If $\alpha > 1$, then $\alpha - 1 > 0$ and $\lim\limits_{\varepsilon\to 0}\varepsilon^{\alpha-1} = 0$. Hence,

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{1-\alpha}}{1-\alpha} = \lim_{\varepsilon\to 0}\frac{1}{(1-\alpha)\varepsilon^{\alpha-1}} = \infty$$

that means the improper integral is divergent.

If $\alpha < 1$, then $1 - \alpha > 0$ and $\lim\limits_{\varepsilon\to 0}\varepsilon^{1-\alpha} = 0$, thus,

$$\lim_{\varepsilon\to 0}\left[\frac{(b-a)^{1-\alpha}}{1-\alpha}-\frac{\varepsilon^{1-\alpha}}{1-\alpha}\right] = \frac{(b-a)^{1-\alpha}}{1-\alpha},$$

i.e. the improper integral is convergent.

If $\alpha = 1$, then

$$\lim_{\varepsilon\to 0}\int\limits_a^{b-\varepsilon}\frac{dx}{(b-x)^\alpha} = -\lim_{\varepsilon\to 0}\int\limits_a^{b-\varepsilon}\frac{d(b-x)}{b-x} = -\lim_{\varepsilon\to 0}\ln|b-x|\Big|_a^{b-\varepsilon} =$$

$$= \lim_{\varepsilon\to 0}(\ln|b-a| - \ln|\varepsilon|) = \infty,$$

i.e. the improper integral is divergent.

Consequently, the improper integral (7.14) is convergent if $\alpha < 1$ja and divergent if $\alpha \geq 1$.

For the improper integrals of unbounded functions there hold the analogous theorems as for the improper integrals over unbounded intervals.

Theorem 3'. If the functions $f(x)$ and $\varphi(x)$ continuous in the half-interval $[a;b)$ satisfy the condition $0 \leq f(x) \leq \varphi(x)$ then the convergence of the improper integral

$$\int\limits_a^b \varphi(x)dx \tag{7.15}$$

yields the convergence of the improper integral

$$\int\limits_a^b f(x)dx \tag{7.16}$$

Theorem 4'. If the functions $f(x)$ and $\varphi(x)$ continuous in the half-interval $[a; b)$ are equivalent in the limiting process $x \to b$ then the convergence of (7.15) yields the convergence of (7.16) and the divergence of (7.15) yields the divergence of (7.16).

Theorem 5'. The absolute convergence of the improper integral (7.16) yields the convergence of that integral.

5.8 The approximate computation of definite integral

Applying the Newton-Leibnitz formula to evaluate the definite integral, we have to find the antiderivative of the integrand. A lot of quite a simple functions, for instance

$$e^{-x^2}, \quad \frac{\sin x}{x} \text{ and } \frac{1}{\ln x}$$

don't have antiderivative among elementary functions. Thus, the Newton-Leibnitz formula is not applicable. In this case we use the approximate formulas to evaluate the definite integral. One of those approximate formulas is called *trapezoidal rule.*

Let us have an integral $\int\limits_a^b f(x)dx$ for a continuous function $f(x) \geq 0$. We divide the interval $[a; b]$ into n subintervals of equal width. So we obtain a partition

$$a = x_0, x_1, \ x_2, \ \dots \ x_{k-1}, \ x_k, \ \dots, \ x_n = b$$

where the length of any subdivision $[x_{k-1}; x_k]$ is

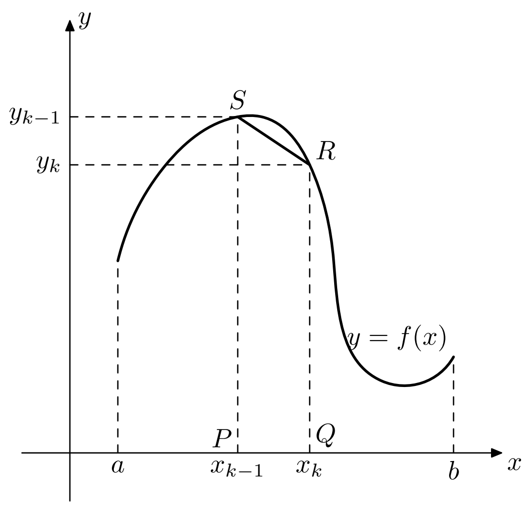
$$h = \frac{b-a}{n}$$

Hence, $x_k - x_{k-1} = h$ for any $k = 1, \ 2, \ \dots, \ n$ and the dividing points are $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ..., $x_k = a + kh$, ..., $x_n = a + nh = b$.

The vertical lines $x = x_k$, $k = 1, \ 2, \ ..., \ n-1$ divide the area $abBA$ under the graph into n areas $PQRS$. If we substitute the curve between R and S by the straight line RS, we obtain the trapezoid $PQRS$, whose parallel sides PS and QR have the lengths $f(x_{k-1})$ and $f(x_k)$, respectively. The length of one subdivision h is the height of trapezoid $PQRS$ and the area of this trapezoid is

$$S_k = \frac{f(x_{k-1}) + f(x_k)}{2} \cdot h$$

The sum of the areas of n trapezoids $PQRS$ equals approximately to the area under the graph $abBA$. If n is increasing, then the accuracy of this



approximation becomes higher. The area under the graph is the value of the definite integral. Thus, the definite integral equals approximately to the sum of the areas of trapezoids $PQRS$:

$$\int_a^b f(x)dx \approx S_1+S_2+\ldots+S_n = \frac{f(x_0)+f(x_1)}{2}\cdot h+\frac{f(x_1)+f(x_2)}{2}\cdot h+\ldots+\frac{f(_{n-1})+f(x_n)}{2}\cdot h$$

Factoring out $\frac{h}{2}$, we have the approximate formula

$$\int_a^b f(x)dx \approx \frac{h}{2}(f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)) \quad (8.17)$$

which is called *trapezoidal rule*. Notice that all the values of the function are multiplied by 2, except the values at the endpoints $y_0 = f(a)$ and $y_n = f(b)$.

Example 12. Compute by trapezoidal rule $\int_0^2 x^2dx$.

To compare the result with exact value, we calculate first this definite integral by Newton-Leibnitz formula $\int_0^2 x^2dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3} = 2,(6)$.

Now we compute this definite integral by trapezoidal formula. First we divide the interval of integration into four four equal parts $[0;2]$, that means $n = 4$. The length of one subdivision is $h = \frac{2-0}{4} = 0,5$ and dividing points are $x_0 = 0$, $x_1 = 0,5$, $x_2 = 1$, $x_3 = 1,5$ and $x_4 = 2$.

Evaluating the function $f(x) = x^2$ at these points, we have $f(x_0) = 0$, $f(x_1) = 0,25$, $f(x_2) = 1$, $f(x_3) = 2,25$ and $f(x_4) = 4$. By trapezoidal rule (8.17)

$$\int_0^2 x^2 dx \approx 0,25(0 + 2 \cdot 0,25 + 2 \cdot 1 + 2 \cdot 2,25 + 4) = 2,75$$

Next we compute this integral by trapezoidal rule again, dividing the interval of integration $[0; 2]$ into eight equal parts. Then the length of one subdivision is $h = 0,25$ and the dividing points are $x_0 = 0$, $x_1 = 0,25$, $x_2 = 0,5$, $x_3 = 0,75$, $x_4 = 1$, $x_5 = 1,25$, $x_6 = 1,5$, $x_7 = 1,75$ and $x_8 = 2$.

The values of the function $f(x) = x^2$ at these points are $f(x_0) = 0$, $f(x_1) = 0,0625$, $f(x_2) = 0,25$, $f(x_3) = 0,5625$, $f(x_4) = 1$, $f(x_5) = 1,5625$, $f(x_6) = 2,25$, $f(x_7) = 3,0625$ and $f(x_8) = 4$. By trapezoidal rule (8.17)

$$\int_0^2 x^2 dx \approx 0,125(0+2\cdot0,0625+2\cdot0,25+2\cdot0,5625+2\cdot1+2\cdot1,5625+2\cdot2,25+2\cdot3,0625+4) = 2,6875$$