5.9 Area in rectangular coordinates

If $f(x) \geq 0$ on the interval $[a;b]$, then the definite integral $\int\limits_a^b f(x)dx$ equals to the area of the region bounded by the graph of the function $y = f(x)$, the x-axis $y = 0$ and two vertical lines $x = a$ and $x = b$.

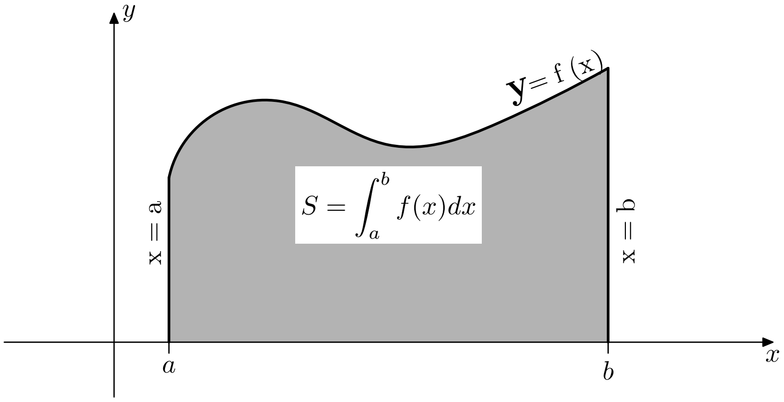

Figure 5.1. the area under the graph of $f(x) \geq 0$

The area under the graph in Figure 5.1 is

$$S_{abBA} = \int\limits_a^b f(x)dx. \tag{5.1}$$

Suppose that the continuous function f has on $[a;b]$ negative values. Consider the computation of the area of the region in Figure 5.2 bounded by vertical lines $x = a$ and $x = b$, the x-axis and the graph of the function $y = f(x)$.

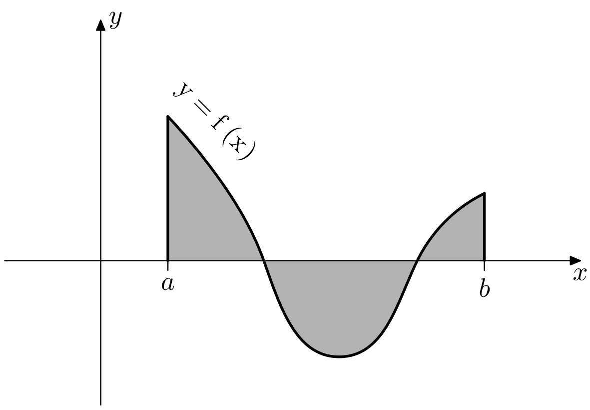

Figure 5.2.

If we substitute the graph of the function $y = f(x)$ by the graph of the function $y = |f(x)|$, then the area of the region bounded by the graph of $y = |f(x)|$, the vertical lines $x = a$ and $x = b$ and x-axis (Figure 5.3) is equal to the area in Figure 5.2.

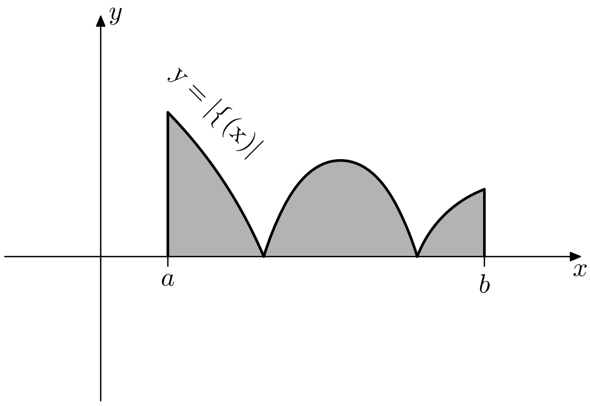

Figure 5.3. The area of the region bounded by the graph of the absolute value of function and x-axis

Since $|f(x)| \geq 0$, the area in Figure 5.3 (thus, the area in Figure 5.2) is by (5.1)

$$S = \int_a^b |f(x)| dx. \tag{5.2}$$

Example 1. Find the area of the region bounded by sinusoid and x-axis, if $x \in [0; 2\pi]$ (Figure 5.4).

By formula (5.2)

$$S = \int_0^{2\pi} |\sin x| dx$$

According to the additivity property of the definite integral

$$S = \int_0^{\pi} |\sin x| dx + \int_{\pi}^{2\pi} |\sin x| dx$$

Since

$$|\sin x| = \begin{cases} \sin x, & \text{if } \sin x \geq 0 \text{ or } x \in [0; \pi] \\ -\sin x, & \text{if } \sin x < 0 \text{ or } x \in (\pi; 2\pi), \end{cases}$$

we obtain

$$\begin{aligned} S &= \int_0^{\pi} \sin x dx - \int_{\pi}^{2\pi} \sin x dx \\ &= -\cos x \Big|_0^{\pi} + \cos x \Big|_{\pi}^{2\pi} = -(-1-1) + 1 - (-1) = 4. \end{aligned}$$

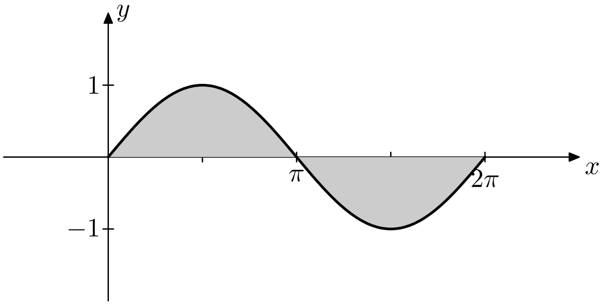

Figure 5.4. The area of the region bounded by sinusoid and x-axis on $[0; 2\pi]$

Next we are going to look at finding the area between two curves. We determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a; b]$ assuming $f(x) \geq g(x)$. The region is drawn in Figure 5.5.

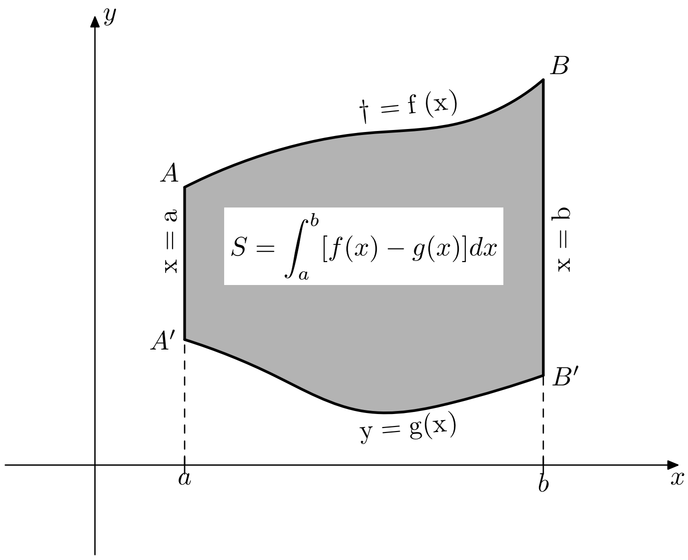

Figure 5.5. The area of the region between two curves

Obviously the area of the region $A'B'BA$ is the difference of areas of $abBA$ and $abB'A'$

$$S_{A'B'BA} = S_{abBA} - S_{abB'A'}$$

By the formula (5.1)

$$S_{A'B'BA} = \int\limits_a^b f(x)dx - \int\limits_a^b g(x)dx$$

and by the property of the definite integral

$$S_{A'B'BA} = \int\limits_a^b [f(x) - g(x)]dx. \tag{5.3}$$

Remark. In Figure 5.5 it has been supposed that $0 \le g(x) \le f(x)$ on $[a;b]$. Actually the request of non-negativity is unnecessary. The formula (5.3) is valid provided $g(x) \le f(x)$ on $[a;b]$.

Example 2. Compute the area of the region bounded by curve $y = \dfrac{1}{1+x^2}$ and parabola $y = \dfrac{x^2}{2}$.

Both functions in this example are even. Therefore, the graphs of these functions are symmetric with respect to y-axis (Figure 5.6).

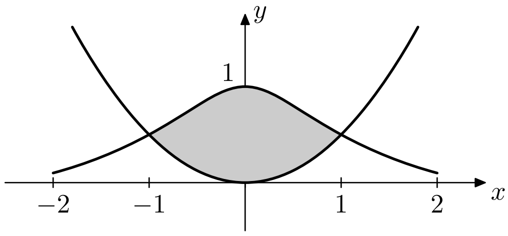

Figure 5.6. The region bounded by the curve $y = \dfrac{1}{1+x^2}$ and parabola $y = \dfrac{x^2}{2}$

To find the abscissas of the points of intersection of these curves we solve the equation

$$\frac{1}{1+x^2} = \frac{x^2}{2}$$

This equation converts to the biquadratic equation $x^4 + x^2 - 2 = 0$, which yields $x^2 = 1$ or $x^2 = -2$. The second equation has no real roots, but the first has two solutions $x_1 = -1$ and $x_2 = 1$. These values of x are the abscissas of the points of intersection of given curves. Thus, the area of the region in Figure 5.6 is by (5.3)

$$S = 2\int\limits_0^1 \left(\frac{1}{1+x^2} - \frac{x^2}{2}\right) dx = 2\left(\arctan x - \frac{x^3}{6}\right)\Big|_0^1 = 2\left(\frac{\pi}{4} - \frac{1}{6}\right) = \frac{\pi}{2} - \frac{1}{3}$$

Further, suppose that the upper function has parametric representation. Consider the region in Figure 5.7.

$$\begin{cases} x = x(t), \\ y = y(t), \end{cases}$$

Suppose that at the point A the value of the parameter is $t = \alpha$ and at the point B $t = \beta$. Then

$$a = x(\alpha) \quad \text{and} \quad b = x(\beta). \tag{5.4}$$

Rewrite the formula (5.1) as

$$S_{abBA} = \int\limits_a^b y dx$$

and change the variable by t. The variable y can be substituted by its parametric representation, the differential of the variable x is $dx = \dot{x}dt$ and the limits of integration for t we get from (5.4). Completing the substitution, we obtain the formula to compute the area of the region $abBA$

$$S_{abBA} = \int\limits_{\alpha}^{\beta} y\dot{x}dt. \tag{5.5}$$

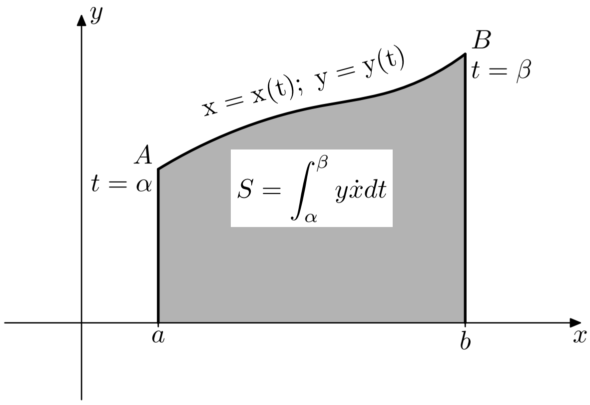

Figure 5.7. The area under the graph of the curve $x = x(t)$, $y = y(t)$

Example 3. Compute the area of the region bounded by ellipse $x = a\cos t$, $y = b\sin t$.

The area bounded by ellipse is in Figure 5.8.

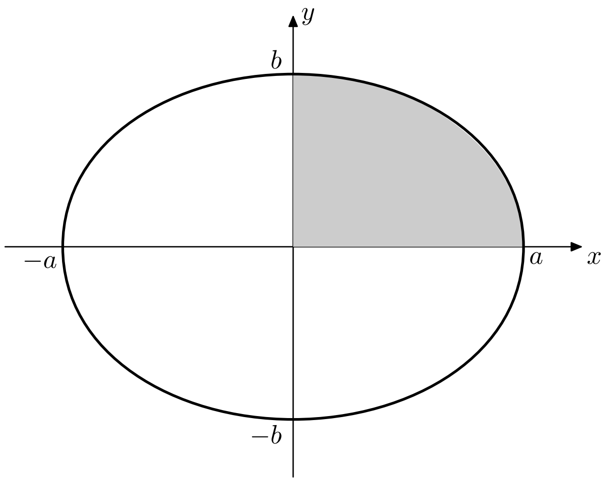

Figure 5.8. The ellipse with semi-axes a and b

The ellipse is centered at the origin and semi-axes are a and b. This ellipse is symmetrical with respect to the both coordinate axes. Therefore, we compute the area under the quarter of this ellipse, which is in the first quadrant of the coordinate plane and multiply the result by 4. At the left endpoint of this quarter $x = 0$ and $y = b$, thus, the parameter $t = \frac{\pi}{2}$, at the right endpoint $x = a$ and $y = 0$, hence, $t = 0$. Since $\dot{x} = -a \sin t$, we obtain by (5.5) the area bounded by ellipse

$$S = 4 \int\limits_{\frac{\pi}{2}}^{0} b \sin t(-a \sin t)dt = -4ab \int\limits_{\frac{\pi}{2}}^{0} \sin^2 t dt$$

Changing the limits of integration and using the formula of sine of half angle, we get

$$\begin{aligned} S &= 2ab \int\limits_{0}^{\frac{\pi}{2}} (1 - \cos 2t)dt = 2ab \int\limits_{0}^{\frac{\pi}{2}} dt - ab \int\limits_{0}^{\frac{\pi}{2}} \cos 2t d(2t) = \\ &= 2abt\Big|_{0}^{\frac{\pi}{2}} - ab \sin 2t\Big|_{0}^{\frac{\pi}{2}} = \pi ab. \end{aligned}$$

5.10 Polar coordinate system. The area in polar coordinates

In mathematics, the *polar coordinate system* is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a fixed point and an angle from a fixed direction.

The fixed point (analogous to the origin of a Cartesian system) is called the *pole*, and the ray from the pole in the fixed direction is the *polar axis*. In mathematical literature, the polar axis is usually drawn horizontal and pointing to the right.

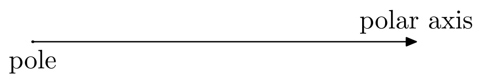

Figure 5.9. Polar coordinate system

The distance from the pole is called the *radial coordinate* or *polar radius*, and the angle is called the *angular coordinate* or *polar angle*. The polar angle is denoted by φ and the polar radius by ρ. Any point P in polar coordinate system is uniquely determined by these two polar coordinates φ and ρ. A positive polar angle means that the angle φ is measured counterclockwise from the polar axis. We say that φ and ρ are the polar coordinates on the point P (Figure 5.10).

If φ is the polar angle of a point, it is obvious that any angle $\varphi \pm 2n\pi$, where n is any integer, is also the polar angle of this point. For a unique representation of any point it is usual to limit φ to the interval $[0; 2\pi)$ or $(-\pi; \pi]$.

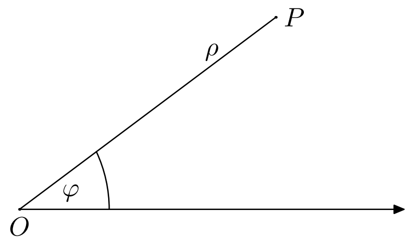

Figure 5.10. The polar coordinates of the point P

To convert the polar coordinates to the Cartesian coordinates we set the cartesian coordinates x and y so that x-axis coincides with polar axis and y-axis passes the pole. Suppose that the polar coordinates of the point P are φ and ρ. From the right triangle OQP (Figure 5.11) we obtain $\cos\varphi = \dfrac{x}{\rho}$

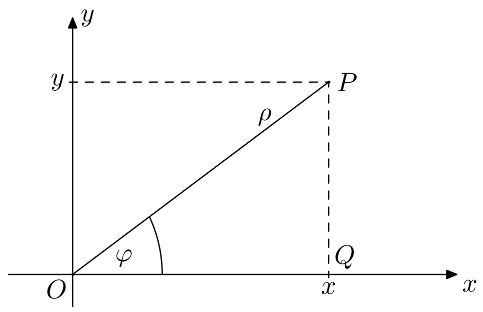

Figure 5.11. Cartesian and polar coordinates

and $\sin\varphi = \dfrac{y}{\rho}$, hence,

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases} \tag{5.6}$$

Squaring the equations (5.6) and adding the results gives

$$x^2 + y^2 = \rho^2\cos^2\varphi + \rho^2\sin^2\varphi$$

or

$$\rho = \sqrt{x^2 + y^2} \tag{5.7}$$

Dividing the second equation of (5.6) by the first, provided $x > 0$, we obtain $\dfrac{y}{x} = \tan\varphi$. The range of the arc tangent function is $\left(-\dfrac{\pi}{2}; \dfrac{\pi}{2}\right)$, but the polar angle has to be in the half-interval $(-\pi; \pi]$. To determine the polar angle uniquely by the Cartesian coordinates x and y, we use the formula

$$\varphi = \begin{cases} \arctan \dfrac{y}{x}, \quad \text{if } x > 0, \\ \arctan \dfrac{y}{x} + \pi, \quad \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan \dfrac{y}{x} - \pi, \quad \text{if } x < 0 \text{ and } y > 0, \\ \dfrac{\pi}{2}, \quad \text{if } x = 0 \text{ and } y > 0, \\ -\dfrac{\pi}{2}, \quad \text{if } x = 0 \text{ and } y < 0, \end{cases} \tag{5.8}$$

The equation defining a curve is in polar coordinates often simpler as the representation in Cartesian coordinates. Such an equation can be specified by defining ρ as a function of φ.

Example 1. Convert the function $(x-r)^2+y^2 = r^2$ to polar coordinates.

The graph of this function is the circle centered at $(r;0)$ and with radius r. Expanding, we get $x^2 - 2rx + r^2 + y^2 = r^2$ or $x^2 + y^2 = 2rx$. Substituting x and y by (5.6), we obtain $\rho^2 = 2r\rho\cos\varphi$ or

$$\rho = 2r\cos\varphi$$

Thus, the polar radius ρ can be expressed as a function of φ, which is quite simple explicit function. The graph of this function is in Figure 5.12.

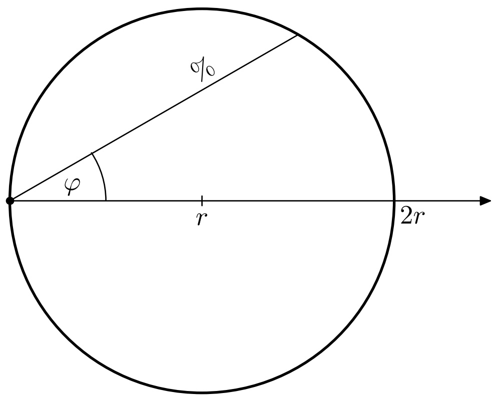

Figure 5.12. The function $\rho = 2r\cos\varphi$

Example 2. Convert the function $(x^2+y^2)^2 = a^2(x^2-y^2)$, where the constant $a > 0$, to polar coordinates.

The drawing of the graph of this function in the Cartesian coordinates is rather complicated. We convert this equation to polar coordinates by (5.6). We obtain $\rho^4 = a^2(\rho^2\cos^2\varphi - \rho^2\sin^2\varphi)$. Dividing this equation by ρ^2 gives $\rho^2 = a^2\cos 2\varphi$ or

$$\rho = a\sqrt{\cos 2\varphi}$$

Again, this function is much more simpler in polar coordinates. Note that the equation is only defined for angles $\cos 2\varphi \geq 0$, i.e. $-\dfrac{\pi}{4} \leq \varphi \leq \dfrac{\pi}{4}$ or

$\frac{3\pi}{4} \leq \varphi \leq \frac{5\pi}{4}$. To draw the graph we find the values of ρ for some values of φ on $-\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}$

φ	0	$\pm\frac{\pi}{12}$	$\pm\frac{\pi}{8}$	$\pm\frac{\pi}{6}$	$\pm\frac{\pi}{4}$
ρ	a	$a\sqrt{\frac{\sqrt{3}}{2}}$	$a\sqrt{\frac{\sqrt{2}}{2}}$	$a\sqrt{\frac{1}{2}}$	0

and on the second interval $\frac{3\pi}{4} \leq \varphi \leq \frac{5\pi}{4}$

φ	π	$\pi \pm \frac{\pi}{12}$	$\pi \pm \frac{\pi}{8}$	$\pi \pm \frac{\pi}{6}$	$\pi \pm \frac{\pi}{4}$
ρ	a	$a\sqrt{\frac{\sqrt{3}}{2}}$	$a\sqrt{\frac{\sqrt{2}}{2}}$	$a\sqrt{\frac{1}{2}}$	0

Substituting the accurate values of ρ by approximate values, we have the points in polar coordinates $(0; a)$, $\left(\pm\frac{\pi}{12}; 0,93a\right)$, $\left(\pm\frac{\pi}{8}; 0,84a\right)$, $\left(\pm\frac{\pi}{6}; 0,71a\right)$, $\left(\pi \pm \frac{\pi}{12}; 0,93a\right)$, $\left(\pi \pm \frac{\pi}{12}; 0,84a\right)$, $\left(\pi \pm \frac{\pi}{12}; 0,71a\right)$ and (π, a). The curve is called *lemniscate of Bernoulli.*

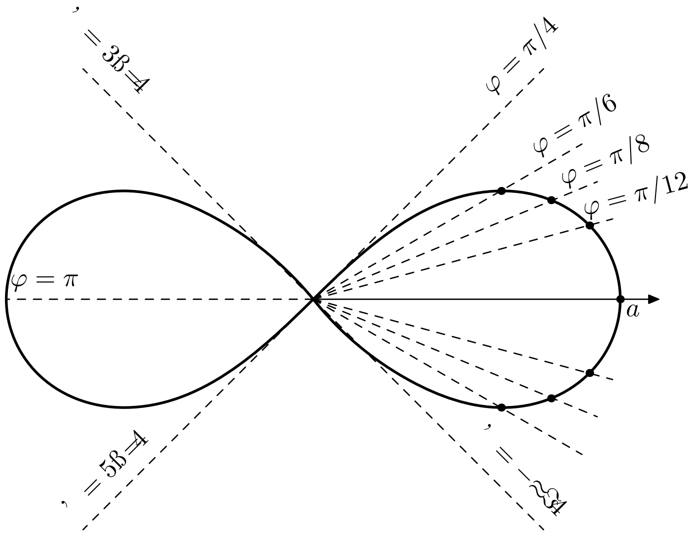

Figure 5.13. Lemniscate of Bernoulli

Let us derive the formula to find the area of the region OAB bounded by straight lines $\varphi = \alpha$, $\varphi = \beta$ and the curve $\rho = f(\varphi)$ (Figure 5.14).

We assume that $\alpha \leq \varphi \leq \beta$ and $\beta \leq \alpha + 2\pi$. Let

$$\alpha = \varphi_0 < \varphi_1 < \varphi_2 < \ldots < \varphi_{k-1} < \varphi_k < \ldots < \varphi_n = \beta$$

be an randomly selected partition of the interval $[\alpha; \beta]$, which divides the interval into n subintervals $[\varphi_{k-1}; \varphi_k]$, where $k = 1, 2, \ldots, n$. Any φ_k is an angle in polar coordinates. In every subinterval we choose a random point $\theta_k \in [\varphi_{k-1}; \varphi_k]$ and approximate the curved sector with central angle $\Delta\varphi_k = \varphi_k - \varphi_{k-1}$ by the sector of the circle OQR with central angle $\Delta\varphi_k$ and radius $f(\theta_k)$ for fixed angle θ_k. In Figure the radius is the length of OP.

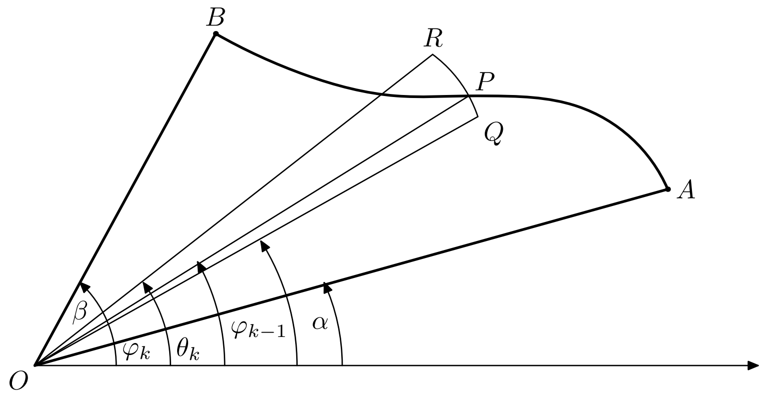

Figure 5.14.

So we have n such sectors of circles. The area of the kth sector is $\dfrac{f^2(\theta_k)\Delta\varphi_k}{2}$. Adding all the areas of these sectors, we obtain the approximate area of region OAB bounded by $\varphi = \alpha$, $\varphi = \beta$ and $\rho = f(\varphi)$

$$\sum_{k=1}^{n} \frac{f^2(\theta_k)}{2} \Delta\varphi_k$$

This sum is the integral sum of the function $\dfrac{f^2(\varphi)}{2}$ on the interval $[\alpha; \beta]$. Denote the greatest length of the subintervals $\lambda = \max\limits_{1 \le k \le n} \Delta\varphi_k$ and consider the limiting process $\lambda \to 0$. It follows that the central angles of all sectors are infinitesimals and the sum of the areas of sectors will represent the area of OAB with greater accuracy. If the function $\rho = f(\varphi)$ is continuous on $[\alpha; \beta]$, then there exists the limit

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} \frac{f^2(\theta_k)}{2} \Delta\varphi_k = \frac{1}{2} \int\limits_{\alpha}^{\beta} f^2(\varphi) d\varphi$$

Consequently, the area of OAB is computed by the formula

$$S = \frac{1}{2} \int\limits_{\alpha}^{\beta} f^2(\varphi) d\varphi \tag{5.9}$$

Example 3. Compute the area of the region bounded by lemniscate of Bernoulli $\rho = a\sqrt{\cos 2\varphi}$.

By Figure 5.13 it is obvious that the lemniscate is symmetrical. It is enough to compute the area of the quarter and multiply the result by 4. We compute the area of the quarter, where $0 \le \varphi \le \dfrac{\pi}{4}$. By the formula (5.9)

$$S = 4 \cdot \frac{1}{2} \int\limits_{0}^{\frac{\pi}{4}} f^2(\varphi) d\varphi = 2 \int\limits_{0}^{\frac{\pi}{4}} a^2 \cos 2\varphi d\varphi = a^2 \int\limits_{0}^{\frac{\pi}{4}} \cos 2\varphi d(2\varphi) = a^2 \sin 2\varphi \Big|_{0}^{\frac{\pi}{4}} = a^2$$

5.11 Length of the arc of curve

Assume that the curve AB is the graph of the continuous on $[a;b]$ function $y = f(x)$ (Figure 5.15) i.e. a is the abscissa of the pint A and b the abscissa of the point B. Assume that the function $f(x)$ has the continuous derivative in the open interval $(a;b)$. In those conditions the curve AB is called *smooth*.

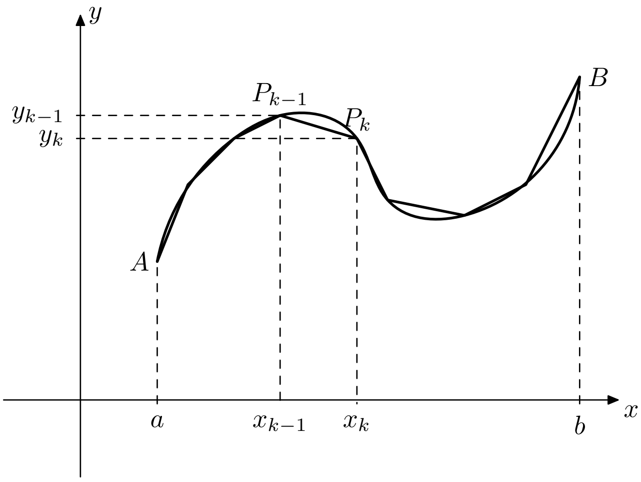

Figure 5.15. The approximation of the curve by a series of straight lines

We choose an arbitrary partition of the curve AB, using the points

$$A = P_0,\ P_1,\ \dots\ P_{k-1},\ P_k,\ \dots P_n = B$$

so that the abscissa x_k of any point is greater than the abscissa x_{k-1} of previous point. Then $\Delta x_k = x_k - x_{k-1} > 0$. We connect the points $P_{k-1}(x_{k-1}; y_{k-1})$ and $P_k(x_k; y_k)$ $(k = 1,\ 2,\ \dots,\ n)$ with straight lines. This creates the broken line $P_0P_1 \dots P_{k-1}P_k \dots P_n$. Denoting the length of the kth line segment by Δs_k, we obtain the length of this broken line

$$\sum_{k=1}^{n} \Delta s_k. \tag{5.10}$$

and this sum is approximately the length of the arc AB.

If the greatest length of line segments $\max\limits_{1 \le k \le n} \Delta s_k \to 0$, then the length of any line segment is approaching zero.

Definition 1. The limit of the length of the broken line, as the greatest length of line segments approaches 0, is called the length of the arc AB and denoted by s, i.e.

$$s = \lim_{\max\limits_{1 \le k \le n} \Delta s_k \to 0} \sum_{k=1}^{n} \Delta s_k. \tag{5.11}$$

Now we derive the formula to compute the length of arc AB using the assumptions. Let $\Delta y_k = y_k - y_{k-1}$. Then the length of the kth line segment is

$$\Delta s_k = \sqrt{\Delta x_k^2 + \Delta y_k^2} = \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \Delta x_k$$

since $\Delta x_k \geq 0$ by construction. The function $f(x)$ satisfies the assumptions of Lagrange theorem. Thus, there exists $\xi_k \in (x_{k-1}, x_k)$ such that

$$\frac{\Delta y_k}{\Delta x_k} = \frac{y_k - y_{k-1}}{x_k - x_{k-1}} = f'(\xi_k)$$

and

$$\Delta s_k = \sqrt{1 + (f'(\xi_k))^2} \Delta x_k$$

and by Definition 1

$$s = \lim_{\max\limits_{1 \leq k \leq n} \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + (f'(\xi_k))^2} \Delta x_k$$

The last sum is the integral sum of the function $\sqrt{1 + [f'(x)]^2}$. Thus, by the definition of the definite integral the length of arc AB is computed by formula

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx \tag{5.12}$$

Example 1. Determine the length of arc of the graph of the function $y = \ln x$ on $x \in [1; \sqrt{3}]$.

Find $y' = \dfrac{1}{x}$, $1 + y'^2 = 1 + \dfrac{1}{x^2}$ and $\sqrt{1 + y'^2} = \dfrac{\sqrt{x^2 + 1}}{x}$. By the formula (5.12)

$$s = \int_1^{\sqrt{3}} \sqrt{x^2 + 1} \cdot \frac{dx}{x}$$

To integrate we change the variable $t = \sqrt{x^2 + 1}$ or $t^2 = x^2 + 1$ and $2tdt = 2xdx$. Dividing both sides by $2x^2$ gives $\dfrac{dx}{x} = \dfrac{tdt}{x^2} = \dfrac{tdt}{t^2 - 1}$. For $x = 1$ $t = \sqrt{2}$ and for $x = \sqrt{3}$ $t = 2$. After substitution

$$\begin{aligned} s &= \int_{\sqrt{2}}^{2} t \cdot \frac{tdt}{t^2 - 1} = \int_{\sqrt{2}}^{2} \frac{t^2 - 1 + 1}{t^2 - 1} dt = \int_{\sqrt{2}}^{2} dt - \int_{\sqrt{2}}^{2} \frac{dt}{1 - t^2} = \\ &= 2 - \sqrt{2} - \frac{1}{2} \ln \left| \frac{1 + t}{1 - t} \right| \Bigg|_{\sqrt{2}}^{2} = 2 - \sqrt{2} - \frac{1}{2} \ln 3 + \frac{1}{2} \ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1} = \\ &= 2 - \sqrt{2} + \frac{1}{2} \ln \frac{(\sqrt{2} + 1)^2}{2 - 1} - \ln \sqrt{3} = 2 - \sqrt{2} + \ln \frac{\sqrt{2} + 1}{\sqrt{3}} \approx 0,918. \end{aligned}$$

Suppose that the curve AB has parametric representation $x = x(t)$ and $y = y(t)$. Let α be the value of the parameter at A and β the value of the parameter at B. Assume that the functions $x = x(t)$ and $y = y(t)$ are continuous on $[\alpha; \beta]$ and have the continuous derivatives in $(\alpha; \beta)$. Also assume that $\dot{x} > 0$, i.e. $x = x(t)$ is strictly increasing in $(\alpha; \beta)$. Change the variable in (5.12) by t. The derivative of the parametric function is $f'(x) = \dfrac{\dot{y}}{\dot{x}}$ and the differential $dx = \dot{x}dt$. If $x = a$, then $t = \alpha$, and if $x = b$, then $t = \beta$. The formula (5.12) converts to

$$s = \int\limits_{\alpha}^{\beta} \sqrt{1 + \left(\frac{\dot{y}}{\dot{x}}\right)^2}\,\dot{x}dt$$

According to assumption $\dot{x} > 0$ we obtain the formula to compute the length of the arc of the curve

$$s = \int\limits_{\alpha}^{\beta} \sqrt{\dot{x}^2 + \dot{y}^2}dt. \tag{5.13}$$

Example 2. Compute the length of one arc of cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

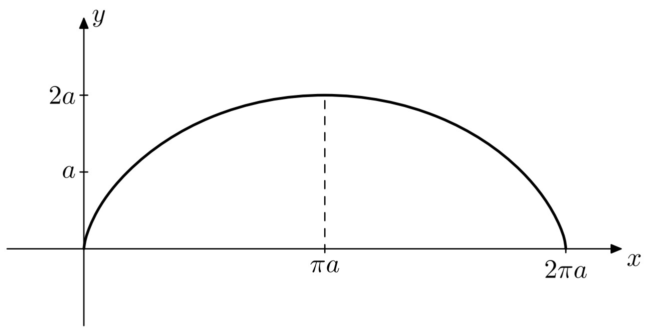

Figure 5.16. The arc of cycloid for $t \in [0; 2\pi]$

The cycloid is a cyclic curve, whose first arc is described by given equations if t changes from 0 to 2π. Find the derivatives with respect to parameter $\dot{x} = a(1 - \cos t)$ and $\dot{y} = a \sin t$. The sum of squares of these derivatives is $\dot{x}^2 + \dot{y}^2 = a^2(1 - \cos t)^2 + a^2 \sin^2 t = a^2(1 - 2\cos t + \cos^2 t + \sin^2 t) = 2a^2(1 - \cos t) = 4a^2 \sin^2 \dfrac{t}{2}$. Consequently, $\sqrt{\dot{x}^2 + \dot{y}^2} = 2a \sin \dfrac{t}{2}$.

Now we obtain by the formula (5.13)

$$s = 2a \int\limits_{0}^{2\pi} \sin \frac{t}{2} dt = 4a \int\limits_{0}^{2\pi} \sin \frac{t}{2} d\left(\frac{t}{2}\right) = 4a\left(-\cos\frac{t}{2}\right)\Big|_{0}^{2\pi} = 8a$$

Remark. The length of the space curve $x = x(t)$, $y = y(t)$ and $z = z(t)$ on $[\alpha; \beta]$ is computed by the formula analogous to (5.13)

$$s = \int\limits_{\alpha}^{\beta} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt. \qquad (5.14)$$

Example 3. Find the length of the first thread of screw line $x = a\cos t$, $y = a\sin t$, $z = bt$, where a and b are positive constants.

The first arc of screw line is determined by the given equations, if $0 \le t \le 2\pi$. To apply the formula (5.14) we find $\dot{x} = -a\sin t$, $\dot{y} = a\cos t$, $\dot{z} = b$ and

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = a^2 + b^2$$

By (5.14) the length of the first thread of screw line is

$$s = \int\limits_{0}^{2\pi} \sqrt{a^2 + b^2} dt = 2\pi\sqrt{a^2 + b^2}.$$

Next let us consider the curve in polar coordinates $\rho = f(\varphi)$, where $\varphi \in [\alpha; \beta]$. Substituting in polar to Cartesian conversion formulas (5.6) the variable ρ by φ, we obtain the parametric equation of a curve

$$x = f(\varphi)\cos\varphi$$
$$y = f(\varphi)\sin\varphi,$$

where the parameter is the polar angle φ.

To derive the formula of the length of arc we apply the formula (5.13). We find $\dot{x} = f'(\varphi)\cos\varphi - f(\varphi)\sin\varphi$ and $\dot{y} = f'(\varphi)\sin\varphi + f(\varphi)\cos\varphi$. Hence,

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 &= f'^2(\varphi)\cos^2\varphi - 2f'(\varphi)\cos\varphi f(\varphi)\sin\varphi + f^2(\varphi)\sin^2\varphi + \\ &+ f'^2(\varphi)\sin^2\varphi + 2f'(\varphi)\sin\varphi f(\varphi)\cos\varphi + f^2(\varphi)\cos^2\varphi = \\ &= f'^2(\varphi)(\cos^2\varphi + \sin^2\varphi) + f^2(\varphi)(\sin^2\varphi + \cos^2\varphi) = f'^2(\varphi) + f^2(\varphi) \end{aligned}$$

Thus, the formula (5.13) gives us the formula to compute the length of arc of the curve in polar coordinates $\rho = f(\varphi)$, where $\alpha \le \varphi \le \beta$

$$s = \int\limits_{\alpha}^{\beta} \sqrt{f^2(\varphi) + f'^2(\varphi)} d\varphi. \qquad (5.15)$$

Example 4. Compute the length of cardioid $\rho = a(1 + \cos\varphi)$ (Figure 5.17).

Since $\cos\varphi$ is an even function, the cardioid is symmetric with respect to polar axis. So we compute the length of the half of cardioid, where $0 \le \varphi \le \pi$ and double the result. To apply the formula (5.15) we find $f'(\varphi) = -a\sin\varphi$ and

$$f^2(\varphi) + f'^2(\varphi) = a^2(1 + \cos\varphi)^2 + a^2\sin^2\varphi = 2a^2(1 + \cos\varphi) = 4a^2\cos^2\frac{\varphi}{2}$$

Now by the formula (5.15)

$$s = 2\int\limits_{0}^{\pi} 2a\cos\frac{\varphi}{2} d\varphi = 8a\int\limits_{0}^{\pi} \cos\frac{\varphi}{2} d\left(\frac{\varphi}{2}\right) = 8a\sin\frac{\varphi}{2}\Big|_{0}^{\pi} = 8a$$

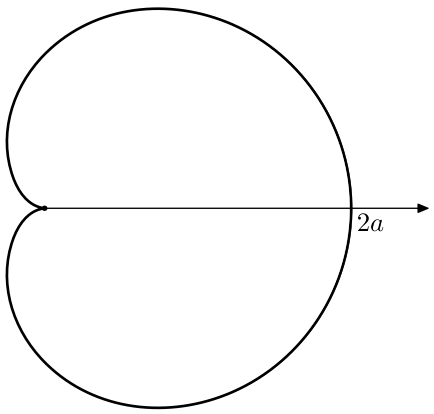

Figure 5.17. Cardioid

5.12 Volumes of revolution

One more application of the definite integral is to find the volume of a solid.

Let the function $f(x)$ continuous on $[a;b]$ satisfy the condition $f(x) \geq 0$. Consider the region (Figure 5.18) $abBA$ bounded by x-axis, straight lines $x = a$, $x = b$ and the graph of the function $y = f(x)$. We rotate this region about x-axis to get the solid of revolution. This gives the following three dimensional region.

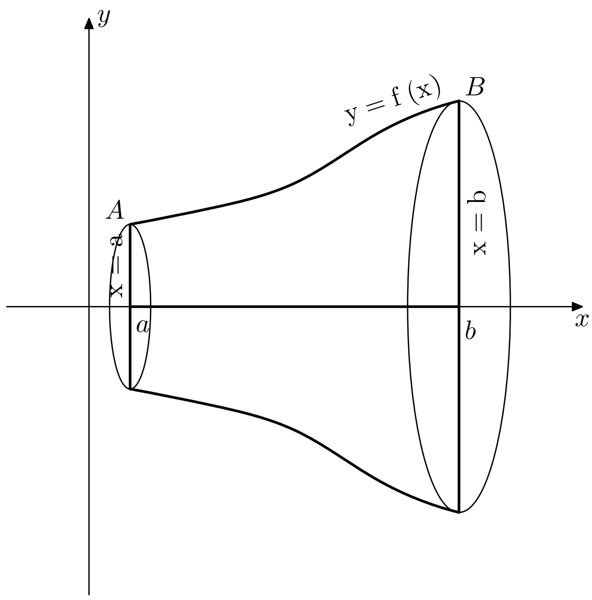

Figure 5.18. The solid obtained by rotating the region $abBA$ about x-axis

What we want to do is to determine the volume of this solid revolution. Let

$$a = x_0 < x_1 < \ldots < x_{k-1} < x_k < \ldots < x_n = b$$

be an arbitrary partition of the interval $[a; b]$. We have n subintervals $[x_{k-1}; x_k]$, $k = 1, 2, \ldots, n$. In every subinterval we choose a random point $\xi_k \in [x_{k-1}; x_k]$. Now we divide the solid of revolution by plains $x = x_k$ ($k = 0, 2, \ldots, n$) perpendicular to x-axis. Next we approximate the volume of the disk between two consequent plains $x = x_{k-1}$ and $x = x_k$ by the volume of cylinder with radius $f(\xi_k)$ and height $\Delta x_k = x_k - x_{k-1}$. The volume of this cylinder is $\Delta v_k = \pi f^2(\xi_k)\Delta x_k$.

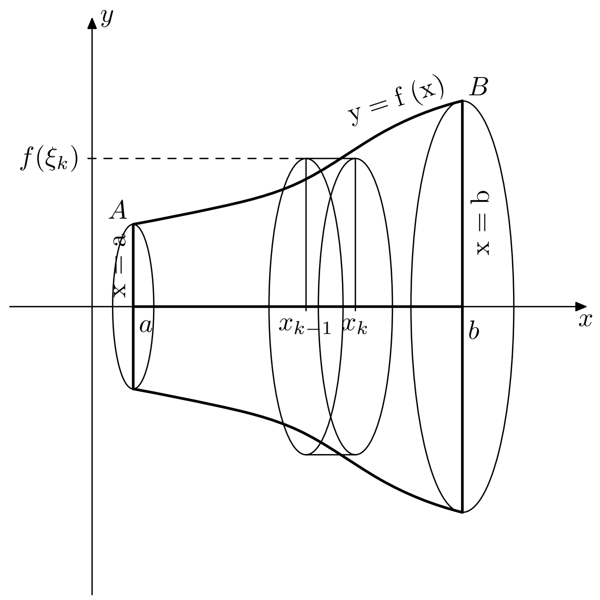

Figure 5.19. Approximation of the solid revolution by the sum of cylinders

The sum of the volumes of these cylinders

$$\sum_{k=1}^{n} \pi f^2(\xi_k)\Delta x_k$$

is the integral sum of the function $\pi f^2(x)$ The limit of this sum as $\max\limits_{1\le k\le n} \Delta x_k \to 0$ equals to the integral $\pi \int\limits_a^b f^2(x)dx$. Thus, the volume of the solid of revolution obtained by rotating the graph of $y = f(x)$, where $x \in [a; b]$ about x-axis

$$V = \pi \int\limits_a^b y^2 dx \tag{5.16}$$

Example 1. Find the volume of solid of revolution obtained by rotation of the half-circle $y = \sqrt{r^2 - x^2}$ about x-axis.

Since the half-circle is symmetrical with respect to y-axis, we find the volume of the solid of revolution obtained by rotation of the quarter of circle

$y = \sqrt{r^2 - x^2}$, where $0 \leq x \leq r$ about x-axis and double the result. Since $y^2 = r^2 - x^2$, we get by the formula (5.16)

$$V = 2\pi \int_0^r (r^2 - x^2)dx = 2\pi \left(r^2 x - \frac{x^3}{3} \right) \Bigg|_0^r = 2\pi \left(r^3 - \frac{r^3}{3} \right) = \frac{4\pi r^3}{3}$$