1 Functions on several variables

To any ordered pair of real numbers (x, y) there is related one point in xy-plane. To any point in xy-plane there are related the coordinates of this point, that means the ordered pair of real numbers. It is said that between ordered pairs of real numbers and the points on xy-plane there is *one-to-one correspondence.*

The subset of the points of xy-plane is called the domain (region). We shall denote the domains by D. For example the domain

$$D = \{(x, y)|\ x^2 + y^2 \leq 1\}$$

is the unit disk centered at the origin, which contains the circle surrounding this disk.

The curve bounding the domain is called the *boundary line* of this domain and the points on the boundary line are called *boundary points*. The points not laying on the boundary line are called *interior points*.

The domain containing all of its boundary points (that means the whole boundary line) is called *closed*.

The domain containing none of its boundary points is called *open* (if it contains some but not all of its boundary points, then it is neither open or closed).

If the domain contains its boundary line or a part of its boundary line, we sketch this line (part of the line) by the continuous line. If the domain does not contain its boundary line or a part of its boundary line, we sketch this line (part of the line) by the dashed line. Any open disk centered at the given point is called the *neighborhood* of this point. If $\delta > 0$ is whatever real number, then the δ-neighborhood of the point(x_0, y_0) is the open disk (without center)

$$U_\delta(x_0, y_0) = \{(x, y)|\ 0 < (x - x_0)^2 + (y - y_0)^2 < \delta^2\}$$

There exists the one-to-one correspondence between the triplets of real numbers (x, y, z) and the points in space. The subset of the (x, y, z)-space is called the *spatial region.*

The spatial region is separated from the rest of the space by a surface, which is called the *boundary surface*. The points on the boundary surface are called the boundary points and the points not laying on the boundary surface are called interior points.

The region is called closed, if it contains all of its boundary points and the region is called open, if it contains none of its boundary points.

Thus, the closed region is the region with the surface surrounding the region and the open region is the region without the surface surrounding the region.

The δ-neighborhood of the spatial point (x_0, y_0, z_0) is the open ball

$$U_\delta(x_0, y_0, z_0) = \{(x, y, z)|\ 0 < (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2\}$$

that means the open ball centered at (x_0, y_0, z_0) with radius δ. This open ball does not contain the sphere surrounding the ball and does not contain the center (x_0, y_0, z_0).

1.1 Functions of two variables

Let D be some domain in the xy-plane (included the whole plane). A function of two variables is a function whose inputs are points $(x; y)$ in the xy-plane and whose outputs real numbers.

Definition 1. If to each point $(x; y) \in D$ there is related one certain value of the variable z, then z is called the function of two variables x and y and denoted

$$z = f(x, y)$$

The function of two variables can be also denoted by $z = g(x, y)$, $z = F(x, y)$, $z = \varphi(x, y)$ or $z = z(x, y)$.

The variables x and y are the *independent variables* and z is the function or the *dependent variable.*

Whenever a quantity depend on two others we have a function of two variables. The area on the rectangle of length x and width y is $S = xy$. The number of items n which can be sold is the function of the price p and the advertising budget a that is $n = f(p, a)$. The force of the suns gravity F depends on an object mass m and the distance d: $F = F(m, d)$.

Further we shall consider the functions given implicitly. In those cases to each point $(x; y) \in D$ there can be related two or more values of the variable z. We talk about the two-valued functions, three-valued functions, etc.

In the graph of the function of two variables $z = f(x, y)$ is the spatial point with coordinates $(x, y, f(x, y))$. The set of all those point is the surface in space. Hence, the graph of the function of two variables $z = f(x, y)$ is the surface in x, y, z-coordinates.

Example 1. The graph of the function $z = 1 - x - y$ is the plane.

Example 2. The graph of the function $z = x^2 + y^2$ is the paraboloid of revolution created by the rotation of the parabola $z = y^2$ around z-axis.

The next surface is the graph of the function of two variables given implicitly.

Example 3. The graph of the function $x^2 + y^2 + z^2 = r^2$ given implicitly is the sphere with radius r centered at the origin.

Solving this equation for the variable z, we obtain two one-valued functions of two variables $z = \sqrt{r^2 - x^2 - y^2}$ and $z = -\sqrt{r^2 - x^2 - y^2}$. The graph of the first function is the upper side and second function the lower side of the sphere.

Definition 2. The domain of the function of two variables $z = f(x, y)$ is the set of ordered pairs (x, y) (the points of the plane) for which by the given rule it is possible to evaluate the value of the function.

Example 4. Let us find the domain of the function $z = \ln(8 - x^2 - y^2) + \sqrt{2y - x^2}$ and sketch it in the coordinate plane.

The function is defined if there hold two conditions

$$\begin{cases} 8 - x^2 - y^2 > 0 \\ 2y - x^2 \geq 0. \end{cases}$$

The first condition yields $x^2 + y^2 < 8$ and the second $y \geq \dfrac{x^2}{2}$. The first condition holds for the points in xy-plane, which belong to the disk centered at the origin and with radius $2\sqrt{2}$. The is no equality to 8, therefore the circle surrounding the disk does no belong to the set and we sketch the circle with dashed line.

The second condition holds for the points in xy-plane, which are above the parabola $y = \dfrac{x^2}{2}$. This condition contains the equality, consequently the parabola belongs to the set and we sketch it with continuous line.

1.2 Plane sections and level curves of graph of function of two variables

To get an idea how does the graph of the function of two variables looks like it is useful to sketch plane sections of this surface by the planes which are perpendicular to one of the coordinate axis (i.e. parallel to one of the coordinate planes). The equation of the yz-plane is $x = 0$, the equation of the xz-plane is $y = 0$ and the equation of the xy-plane is $z = 0$.

The plane $x = a$ is perpendicular to x-axes, i.e. parallel to yz-plane; the plane $y = b$ is perpendicular to y-axes i.e. parallel to xz-plane; the plane $z = c$ is perpendicular to z-axes i.e. parallel to xy-plane.

The intersections of the surface $z = f(x, y)$ with the planes $x = a$ are the curves

$$\begin{cases} z = f(x, y) \\ x = a, \end{cases}$$

The intersections of the surface $z = f(x; y)$ with the planes $y = b$ are the curves

$$\begin{cases} z = f(x, y) \\ y = b \end{cases}$$

The intersections of the surface $z = f(x; y)$ with the planes $z = c$ are the

$$\begin{cases} z = f(x, y) \\ z = c, \end{cases}$$

The projection of the resulting curve onto the xy-plane is called the *level curve.* A collection of level curves of a surface is called a *contour map.*

Example 1. Let us sketch the surface $x^2 + y^2 - z^2 = 0$, using the intersections with the planes $z = 0$, $z = \pm 1$, $z = \pm 2$ and $x = 0$. First five are the horizontal curves and the sixth is the intersection with the yz-plane.

The intersection of this surface with xy-plane $z = 0$ is actually one point determined by the equations $x^2 + y^2 = 0$, $z = 0$, which is the origin.

The intersection of this surface with the horizontal plane $z = 1$ is the circle $x^2 + y^2 = 1$, $z = 1$, the unit circle on the plane $z = 1$ centered at $(0; 0; 1)$.

The intersection of this surface with the horizontal plane $z = -1$ is the unit circle $x^2 + y^2 = 1$ again but centered at $(0; 0; -1)$.

The intersection of this surface with the horizontal plane $z = 2$ is the circle $x^2 + y^2 = 4$ centered at $(0; 0; 2)$ with radius 2.

The intersection of this surface with the horizontal plane $z = -2$ is the circle $x^2 + y^2 = 4$ centered at $(0; 0; -2)$ with radius 2.

The intersection of this surface with the vertical plane $x = 0$ is determined by $z^2 = y^2$, $x = 0$, that is two perpendicular lines $z = y$ and $z = -y$ on yz-

plane. Adding these two lines to our sketch it turns obvious that the given surface is the cone, whose vertex is at the origin.

If we convert the function $x^2+y^2-z^2=0$ to the explicit form we obtain two one-valued functions $z=\sqrt{x^2+y^2}$ and $z=-\sqrt{x^2+y^2}$. The graph of the first function is the upper part of the cone and the graph of the second function is the lower part of the cone.

Example 2. Let us sketch the surface $z=x^2-y^2$, using the intersections with the planes $y=0$, $x=\pm 1$, $x=\pm 0,5$, $x=0$, $z=0$ and $z=-0,44$.

In this example we draw the coordinate axes in an unusual way, taking the sheet of paper the xz-plane and directing y-axes backwards.

The intersection with the plane $y=0$ is the parabola $z=x^2$, $y=0$.

The intersections with the planes $x=\pm 1$ are the parabolas $z=1-y^2$, $x=1$ and $z=1-y^2$, $x=-1$.

The intersections with the planes $x=\pm 0,5$ are the parabolas $z=0,25-y^2$, $x=0,5$ and$z=0,25-y^2$, $x=-0,5$.

The intersections with the plane $z=0$ are two perpendicular lines $y=x$ and $y=-x$ on the xy-plane.

The intersection with the plane $z=-0,44$ is the equilateral hyperbola $y^2-x^2=0,44$, whose real axis is the y-axis.

The *level surfaces* of the graph of function of three variables $w=f(x,y,z)$ are the surfaces

$$\begin{cases} w=f(x,y,z) \\ w=c. \end{cases}$$

This system of equations yields the equation $f(x,y,z)=c$, the function of two variables given implicitly, whose graph is a surface in the space.

Example 3. The level surfaces of the function $w=x^2+y^2+z^2$ are $x^2+y^2+z^2=c$ provided $c>0$. Those surfaces are the spheres centered at the origin with radius $\sqrt{c}$.

1.3 Increment of function of several variables

Let us fix one point $P(x,y)$ in the domain of the function $z=f(x,y)$. Changing the variable x by Δx and y by Δy, we obtain a point $Q(x+\Delta x, y+\Delta y)$. Assuming that the increments of the independent variables Δx and Δy are sufficiently small, that is Q is also in the domain of the function, we define the *total increment* of the function

$$\Delta z=f(x+\Delta x, y+\Delta y)-f(x,y) \tag{1.1}$$

Assuming that y is a constant or $\Delta y=0$, we have the increment of the function with respect to variable x.

$$\Delta_x z = f(x + \Delta x, y) - f(x, y) \quad (1.2)$$

Assuming that x is a constant or $\Delta x = 0$, we have the increment of the function with respect to variable y.

$$\Delta_y z = f(x, y + \Delta y) - f(x, y) \quad (1.3)$$

One might guess that $\Delta z = \Delta_x z + \Delta_y z$ but as the following example proves, this is not true.

Example 1. For the function $z = xy$ let us find Δz and $\Delta_x z + \Delta_y z$ if $x = 2$, $y = 3$, $\Delta x = 0,2$ and $\Delta y = 0,1$.

First $\Delta_x z = (x + \Delta x)y - xy = y\Delta x = 3 \cdot 0,2 = 0,6$,

second $\Delta_y z = x(y + \Delta y) - xy = x\Delta y = 2 \cdot 0,1 = 0,2$. Thus, $\Delta_x z + \Delta_y z = 0,8$.

The total increment of the function $\Delta z = (x + \Delta x)(y + \Delta y) - xy = x\Delta y + y\Delta x + \Delta x\Delta y = 2 \cdot 0,1 + 3 \cdot 0,2 + 0,2 \cdot 0,1 = 0,82$.

The total increment of the function of three variables $w = f(x, y, z)$ is defined as

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

If y and z are constants, we define

$$\Delta_x w = f(x + \Delta x, y, z) - f(x, y, z)$$

if x and z are constants, we define

$$\Delta_y w = f(x, y + \Delta y, z) - f(x, y, z)$$

and if x and y are constants, we define

$$\Delta_z w = f(x, y, z + \Delta z) - f(x, y, z)$$

1.4 Limit and continuity of functions of two variables

Suppose $P_0(x_0, y_0)$ is a fixed point in the domain of the function $z = f(x, y)$ and $P(x, y)$ is a moving point that approaches P_0. We shall write $(x, y) \to (x_0, y_0)$ or $x \to x_0$, $y \to y_0$.

To find the limit of a function of one variable, we only needed to test the approach from the left and the approach from the right. If both approaches gave the same result, the function had a limit. To find the limit of a function of two variables however, we must show that the limit is the same no matter from which direction we approach (x_0, y_0)

The moving point P can approach the fixed point P_0 along whatever path: along the straight line, broken line, the arc of parabola etc. Independently of the path, the moving point P reaches to any neighborhood of $U_\delta(x_0, y_0)$ for arbitrary small $\delta > 0$.

Definition 1. The real number L is called the limit of the function $f(x, y)$ in limiting process $(x, y) \to (x_0, y_0)$, if $\forall\, \varepsilon > 0$ there exists the neighborhood $U_\delta(x_0, y_0)$ such that $|f(x, y) - L| < \varepsilon$ whenever $(x, y) \in U_\delta(x_0, y_0)$

In other words, L is the limit of the function $f(x, y)$ as $(x, y) \to (x_0, y_0)$, if the value of the function $f(x, y)$ can be made as close as desired to L by taking $P(x, y)$ in the neighborhood of $P_0(x_0, y_0)$ small enough.

This is denoted

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

Example 1. Find the limit $\displaystyle\lim_{(x,y)\to(0;0)} \frac{xy}{x^2 + y^2}$.

Let $(x, y) \to (0; 0)$ along the line $y = kx$. Then

$$\frac{xy}{x^2 + y^2} = \frac{x \cdot kx}{x^2 + k^2x^2} = \frac{k \cdot x^2}{x^2(1 + k^2)} = \frac{k}{1 + k^2}$$

This shows that the result depends on the choice of the slope of the line k. Therefore, the limit does not exist.

Often it is useful to convert the limit into polar coordinates, taking $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$. Then $x^2 + y^2 = \rho^2$ and the limiting process $(x, y) \to (0; 0)$ is equivalent to $\rho \to 0$. In Example 1 we could write

$$\lim_{(x,y)\to(0;0)} \frac{xy}{x^2 + y^2} = \lim_{\rho\to 0} \frac{\rho\cos\varphi\rho\sin\varphi}{\rho^2\cos^2\varphi + \rho^2\sin^2\varphi} = \lim_{\rho\to 0} \frac{\rho^2\cos\varphi\sin\varphi}{\rho^2} = \cos\varphi\sin\varphi$$

The result depends on the polar angle and this proves again that the limit does not exist.

Example 2. Find the limit $\displaystyle\lim_{(x,y)\to(0;0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$.

Converting this limit into polar coordinates, we have

$$\lim_{(x,y)\to(0;0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{\rho\to 0} \frac{\sin(\rho^2\cos^2\varphi + \rho^2\sin^2\varphi)}{\rho^2\cos^2\varphi + \rho^2\sin^2\varphi} = \lim_{\rho\to 0} \frac{\sin\rho^2}{\rho^2} = 1$$

Definition 2. The function $f(x, y)$ is called continuous at the point $P_0(x_0, y_0)$, if

1. $\exists f(x_0, y_0)$

2. $\exists \displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)$

3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Definition 3. The function is called continuous in the domain D, if it is continuous at every point of this domain.

Let us denote the fixed point in Definition 2 by (x, y) and the moving point by $(x + \Delta x, y + \Delta y)$. Then $(x + \Delta x, y + \Delta y) \to (x, y)$ if and only if $(\Delta x, \Delta y) \to (0; 0)$. The third condition of continuity can be re-written

$$\lim_{(\Delta x,\Delta y)\to(0;0)} f(x + \Delta x, y + \Delta y) = f(x, y)$$

or

$$\lim_{(\Delta x,\Delta y)\to(0;0)} [f(x + \Delta x, y + \Delta y) - f(x, y)] = 0 \tag{1.4}$$

In square brackets of the last condition there is the total increment Δz of the function $z = f(x, y)$ and the condition of the continuity (3.42) at the point (x, y) is

$$\lim_{(\Delta x,\Delta y)\to(0;0)} \Delta z = 0 \tag{1.5}$$

The equality (1.5) is called the **necessary and sufficient condition of continuity**.

1.5 Partial derivatives

Fix in the domain of the function of two variables $z = f(x, y)$ one point $P(x, y)$. Holding y constant and increasing the variable x by Δx we have the increment of the function $f(x, y)$

$$\Delta_x z = f(x + \Delta x, y) - f(x, y)$$

.

Definition 1. If there exists the limit

$$\frac{\partial z}{\partial x} = \lim_{\Delta x\to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \tag{1.6}$$

then this limit is called the *partial derivative* of the function $f(x, y)$ with respect to the variable x at the point (x, y).

The partial derivative with respect to x is denoted also z'_x, $f'_x(x, y)$, $\dfrac{\partial f}{\partial x}$.

Holding x constant and increasing the variable y by Δy we have the increment of the function $f(x, y)$ as $\Delta_y z = f(x, y + \Delta y) - f(x, y)$.

Definition 2. If there exists the limit

$$\frac{\partial z}{\partial y} = \lim_{\Delta y\to 0} \frac{\Delta_y z}{\Delta y} = \lim_{\Delta y\to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \tag{1.7}$$

then this limit is called $f(x, y)$ the partial derivative of the function $f(x, y)$ with respect to the variable y at the point (x, y).

The possible alternate notations for partial derivatives with respect to y are z'_y, $f'_y(x, y)$, $\dfrac{\partial f}{\partial y}$.

If we find the partial derivative with respect to the variable x the variable y is treated as constant. The only variable in Definition 1 is Δx. As well, finding the partial derivative with respect to the variable y the variable x is treated as constant. The only variable in Definition 2 is Δy. We need to pay very close attention to which variable we are differentiating with respect to. This is important because we are going to treat the other variable as constant and then proceed with the derivative as if it was a function of a single variable. Consequently, all the rules of differentiation of functions of one variable hold if we find the partial derivatives.

Example 1. Find the partial derivatives with respect to both variables for the function $z = x^3y - x^2y^2$.

Finding the partial derivative with respect to x, y is treated as constant. Thus, by the difference rule an constant rule we obtain

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3y) - \frac{\partial}{\partial x}(x^2y^2) = y\frac{\partial}{\partial x}(x^3) - y^2\frac{\partial}{\partial x}(x^2) = y \cdot 3x^2 - y^2 \cdot 2x = 3x^2y - 2xy^2.$$

Finding the partial derivative with respect to y, x is treated as constant. By the rules of differentiation

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3y) - \frac{\partial}{\partial y}(x^2y^2) = x^3\frac{\partial}{\partial y}(y) - x^2\frac{\partial}{\partial y}(y^2) = x^3 - x^2 \cdot 2y = x^3 - 2x^2y$$

The chain rule is also still valid.

Example 2. Find the partial derivatives with respect to both variables for the function $z = \arctan\dfrac{x}{y}$.

The partial derivative with respect to x is

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \left(\dfrac{x}{y}\right)^2} \cdot \frac{\partial z}{\partial x}\left(\frac{x}{y}\right) = \frac{y^2}{y^2 + x^2} \cdot \frac{1}{y}\frac{\partial}{\partial x}(x) = \frac{y}{x^2 + y^2}$$

The partial derivative with respect to y is

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{1}{1 + \left(\dfrac{x}{y}\right)^2} \cdot \frac{\partial z}{\partial y}\left(\frac{x}{y}\right) = \frac{y^2}{y^2 + x^2} \cdot x\frac{\partial}{\partial y}\left(\frac{1}{y}\right) \\ &= \frac{y^2}{x^2 + y^2} \cdot \left(-\frac{x}{y^2}\right) = -\frac{x}{x^2 + y^2}\end{aligned}$$

The partial derivatives of the function of three variables $w = f(x, y, z)$ with respect to variables x, y and z are defined as the limits

$$\frac{\partial w}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta_x w}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = \lim_{\Delta y \to 0} \frac{\Delta_y w}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

and

$$\frac{\partial w}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta_z w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

If we find the partial derivative with respect to one independent variable, the other independent variables are treated as constants.

Example 3. Find the partial derivatives with respect to all three independent variables for the function $w = x^{y^z}$.

Finding the partial derivative with respect to x, we have the power function with constant exponent y^z, therefore,

$$\frac{\partial w}{\partial x} = y^z x^{y^z - 1}$$

To find the partial derivative with respect to y we use the chain rule. The outside function is the exponential function with constant base x and the variable exponent y^z, which is the power function with respect to y. By the chain rule

$$\frac{\partial w}{\partial y} = x^{y^z} \ln x \cdot z y^{z-1}$$

To find the partial derivative with respect to z we use the chain rule again. The outside function is the exponential function with constant base x. The inside function is another exponential function y^z with the constant base y. Thus

$$\frac{\partial w}{\partial z} = x^{y^z} \ln x \cdot y^z \ln y$$

1.6 Total increment and total differential

Let us fix one point $P(x, y)$ in the domain of function $z = f(x, y)$. Assume that the function $f(x, y)$ is continuous and has the continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $P(x, y)$ and in some neighborhood of this point.

It is possible to prove that total increment (1.1) can be represented as

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \tag{1.8}$$

where ε_1 and ε_2 are two infinitesimals as $(\Delta x; \Delta y) \to (0; 0)$ i.e.

$$\lim_{(\Delta x;\Delta y)\to(0;0)} \varepsilon_1 = \lim_{(\Delta x;\Delta y)\to(0;0)} \varepsilon_2 = 0$$

In subsection 1.4 we have used the notation $\Delta\rho = \sqrt{\Delta x^2 + \Delta y^2}$. The conditions

$$\left|\frac{\Delta x}{\Delta \rho}\right| \leq 1$$

and

$$\left|\frac{\Delta y}{\Delta \rho}\right| \leq 1$$

mean that these are the bounded quantities. Thus, $\varepsilon_1 \dfrac{\Delta x}{\Delta \rho}$ and $\varepsilon_2 \dfrac{\Delta y}{\Delta \rho}$ are infinitesimals as the products of the infinitesimals and a bounded quantities. Thus, the limit

$$\lim_{\Delta\rho\to 0} \frac{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}{\Delta \rho} = \lim_{\Delta\rho\to 0} \varepsilon_1 \frac{\Delta x}{\Delta \rho} + \lim_{\Delta\rho\to 0} \varepsilon_2 \frac{\Delta y}{\Delta \rho} = 0$$

which means that $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ is an infinitesimal of the higher order with respect to $\Delta\rho$, i.e. with respect to Δx and Δy.

After that in the representation (1.8) $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are the values of partial derivatives at the fixed point P i.e. the real numbers. Hence, the first sum

$$\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \tag{1.9}$$

is linear with respect to Δx and Δy.

Definition. The linear part (1.9) of the total increment (1.8) is called the *total differential* of the function $z = f(x, y)$ and denoted by dz.

According to the definition

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

For the function $z = x$ the partial derivatives $\dfrac{\partial z}{\partial x} = 1$, $\dfrac{\partial z}{\partial y} = 0$ and $dz = dx = \Delta x$.

For the function $z = y$ the partial derivatives $\dfrac{\partial z}{\partial x} = 0$, $\dfrac{\partial z}{\partial y} = 1$ and $dz = dy = \Delta y$.

Consequently for the independent variables x and y the notions of differential and increment coincide and the total differential can be re-written as

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy. \tag{1.10}$$

Example 1. Find the total differential for the function $z = \arctan\frac{x}{y}$. Using the partial derivatives found in Example 2 of subsection 1.5, we obtain

$$dz = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy = \frac{ydx - xdy}{x^2+y^2}$$

Example 2. Evaluate the total increment and the total differential for the function $z = \sqrt{x^2+y^2}$, if $x = 3$, $y = 4$, $\Delta x = 0,2$ and $\Delta y = 0,1$.

By the formula of the total increment of the function (1.1) we get

$$\Delta z = \sqrt{3,2^2 + 4,1^2} - \sqrt{3^2+4^4} = \sqrt{27,05} - \sqrt{25} = 0,20096$$

To evaluate the total differential we find

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

and

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

Then

$$dz = \frac{3}{\sqrt{3^2+4^2}} \cdot 0,2 + \frac{4}{\sqrt{3^2+4^2}} \cdot 0,1 = \frac{0,6}{5} + \frac{0,4}{5} = 0,2$$

We see that the difference between the total increment and the total differential is less than $0,001$, which is less by two orders of values with respect to Δx and Δy.

The last fact gives us the possibility to compute the approximate values of functions of two variables using the total differential. If Δx and Δy are sufficiently small, then Δz and dz differ by the quantity, which is the infinitesimal of a higher order with respect to Δx and Δy. We can write

$$\Delta z \approx dz$$

or

$$f(x+\Delta x, y+\Delta y) - f(x,y) \approx \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

This gives us the formula of approximate computation

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \tag{1.11}$$

Example 3. Using the total differential, compute $2,03^3 \cdot 0,96^2$.

Here we choose the function $f(x,y) = x^3y^2$ and the values $x = 2$, $y = 1$, $\Delta x = 0,03$ and $\Delta y = -0,04$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 3x^2y^2$$

and

$$\frac{\partial f}{\partial y} = 2x^3y$$

The value of the function at the point chosen $f(2,1) = 8 \cdot 1 = 8$ and the values of partial derivatives are $\frac{\partial f}{\partial x} = 3 \cdot 4 \cdot 1 = 12$ and $\frac{\partial f}{\partial y} = 2 \cdot 8 \cdot 1 = 16$. By the formula (1.14)

$$(2+0,03)^3 \cdot (1-0,04)^2 = 8 + 12 \cdot 0,03 - 16 \cdot 0,04 = 7,72$$

Suppose that the function of three variables $w = f(x,y,z)$ and the partial derivatives $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$ are continuous at the point $P(x,y,z)$ and in some neighborhood of this point. Analogously to the formula (1.8) it is possible to prove that the total increment of the function can be expressed as

$$\Delta w = \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z + \alpha\Delta x + \beta\Delta y + \gamma\Delta z, \tag{1.12}$$

where $\alpha\Delta x + \beta\Delta y + \gamma\Delta z$ is an infinitesimal of a higher order with respect to $\Delta\rho = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. The expression

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \tag{1.13}$$

is called the total differential of the function $w = f(x,y,z)$. Again, for the independent variables x, y and z the notions of the increment and differential coincide, i.e. $dx = \Delta x$, $dy = \Delta y$ and $dz = \Delta z$.

Example 4. Find the total differential for the function $w = x^{y^z}$.

Using the partial derivatives found in Example 3 of subsection 1.5, we obtain

$$\begin{aligned} dw &= y^z x^{y^z-1}dx + x^{y^z}\ln x \cdot zy^{z-1}dy + x^{y^z}\ln x \cdot y^z \ln y = \\ &= y^z x^{y^z}\left(\frac{dx}{x} + \frac{z\ln x dy}{y} + \ln x \ln y\right) \end{aligned}$$

As well as for the function of two variables there holds the formula of approximate computation

$$f(x+\Delta x, y+\Delta y, z+\Delta z) \approx f(x,y,z) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z \quad (1.14)$$

1.7 Partial derivatives of implicit function

Consider the function

$$F(x,y) = 0 \quad (1.15)$$

given implicitly. This equation determines the variable y as the function of x (in general case not one-valued).

Suppose that the function $F(x,y)$ is continuous and it has the continuous partial derivatives at the point $P(x,y)$ and in some neighborhood of this point. In addition suppose that at $P(x,y)$ the partial derivative $F'_y(x,y) \neq 0$. Let us deduce the formula to find the derivative $\frac{dy}{dx}$, using the partial derivatives of the function $F(x,y)$.

Let us fix the point $P(x,y)$ on the graph of given function. The coordinates of this point satisfy the equation (3.38). Change x by Δx and find on the graph the value of $y+\Delta y$ related to $x+\Delta x$. As $Q(x+\Delta x, y+\Delta y)$ is a point on the graph again, the coordinates of this point also satisfy the equation

$$F(x+\Delta x, y+\Delta y) = 0. \quad (1.16)$$

Subtracting from the equation (1.16) the equation (3.38), we obtain

$$F(x+\Delta x, y+\Delta y) - F(x,y) = 0$$

The left side of the last equality is the total increment of the function $F(x,y)$ and the equality can be re-written

$$\Delta F = 0$$

Because of the assumptions made in the beginning of this subsection this equality converts by (1.8) to

$$\frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y + \alpha\Delta x + \beta\Delta y = 0$$

which yields

$$\left(\frac{\partial F}{\partial y} + \beta\right) \Delta y = -\left(\frac{\partial F}{\partial x} + \alpha\right) \Delta x$$

or

$$\frac{\Delta y}{\Delta x} = -\frac{\dfrac{\partial F}{\partial x} + \alpha}{\dfrac{\partial F}{\partial y} + \beta}$$

Find the limits of both sides of this equality as $\Delta x \to 0$. The limit of the left side is by the definition of the derivative $\dfrac{dy}{dx}$. The function is continuous, consequently if $\Delta x \to 0$ then $\Delta y \to 0$. Knowing that α and β are the infinitesimals as $(\Delta x, \Delta y) \to (0; 0)$, that is $\lim\limits_{\Delta x \to 0} \alpha = 0$ and $\lim\limits_{\Delta x \to 0} \beta = 0$, the limit of the right side of the equality is

$$-\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}}$$

Thus, to find the derivative of the function given implicitly we have the formula

$$\frac{dy}{dx} = -\frac{F'_x}{F'_y} \tag{1.17}$$

Example 1. Find $\dfrac{dy}{dx}$ for $x^4 + y^4 - a^2x^2y^2 = 0$.

Here $F(x, y) = x^4 + y^4 - a^2x^2y^2$, so $F'_x = 4x^3 - 2a^2xy^2$ and $F'_y = 4y^3 - 2a^2x^2y$. By the formula (1.17)

$$\frac{dy}{dx} = -\frac{4x^3 - 2a^2xy^2}{4y^3 - 2a^2x^2y} = -\frac{x(2x^2 - a^2y^2)}{y(2y^2 - a^2x^2)}.$$

The equation $F(x, y, z) = 0$ relates to pairs of (x, y) some value(s) of the variable z. In other words, this equation defines z as a function of x and y. Assume that the function $F(x, y, z)$ is continuous and has the continuous partial derivatives $\dfrac{\partial F}{\partial x}$, $\dfrac{\partial F}{\partial y}$ and $\dfrac{\partial F}{\partial z}$ at the point $P(x, y, z)$ and in some neighborhood of this point. Moreover assume that $F'_z(x, y, z) \neq 0$ at $P(x, y, z)$.

If we find the partial derivative of the function z with respect to x the variable y is treated as constant. In this case in the equation $F(x, y, z) = 0$ there are only two variables x and z and by (1.17) we obtain

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} \tag{1.18}$$

If we repeat this reasoning for y we have

$$\frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z} \tag{1.19}$$

Example 2. Find the partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the function of two variables $x^2 + y^2 + z^2 = r^2$ given implicitly.

As $F'_x = 2x$, $F'_y = 2y$ and $F'_z = 2z$ we obtain by the formula (1.18) the partial derivative

$$\frac{\partial z}{\partial x} = -\frac{x}{z}$$

and by the formula (1.19) the partial derivative

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

1.8 Partial derivatives of composite functions

Suppose that the variable z is a function of two variables u and v, denote $z = f(u, v)$, and u and v are the functions of two independent variables x and y, denote $u = \varphi(x, y)$ and $v = \psi(x, y)$. Then z is a composite function with respect to x and y, i.e.

$$z = f(\varphi(x, y), \psi(x, y)) = F(x, y)$$

Let us fix a point $P(x, y)$ in the common domain of the functions $u = \varphi(x, y)$ and $v = \psi(x, y)$. Then the related point (u, v) in the (u, v)-plane is also fixed. Suppose that the functions u and v are continuous and have the continuous partial derivatives $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ at the point $P(x, y)$ and in some neighborhood of this point. Also assume that the function z is continuous and has the continuous partial derivatives $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ at the related point (u, v) and in some neighborhood of this point.

The partial derivative of the composite function $z = F(x, y)$ with respect to x will be found by the formula

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \tag{1.20}$$

The partial derivative of the composite function z with respect to the variable y will be found by

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \tag{1.21}$$

Example 1. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = \ln(u^2+v)$, $u = e^{x+y^2}$ and $v = x^2+y$.

According to the formulas (1.20) and (1.21) we have to find six partial derivatives

$$\frac{\partial z}{\partial u} = \frac{2u}{u^2+v}, \quad \frac{\partial z}{\partial v} = \frac{1}{u^2+v};$$

$$\frac{\partial u}{\partial x} = e^{x+y^2}, \quad \frac{\partial u}{\partial y} = 2ye^{x+y^2};$$

$$\frac{\partial v}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 1;$$

By (1.20) we have

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2+v}e^{x+y^2} + \frac{1}{u^2+v}2x = \frac{2}{u^2+v}(ue^{x+y^2} + x)$$

and by (1.21)

$$\frac{\partial z}{\partial y} = \frac{2u}{u^2+v}2ye^{x+y^2} + \frac{1}{u^2+v} = \frac{1}{u^2+v}(4uye^{x+y^2} + 1)$$

Remark. If z is a function of three variables $z = f(u, v, w)$ and in addition to the u and v there is $w = \chi(x, y)$, then the partial derivatives of the composite function z with respect to the variables x and y can be found by the formulas

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial x} \tag{1.22}$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial y} \tag{1.23}$$

Next, let z be a function of three variables x, u and v $z = f(x, u, v)$, where $u = \varphi(x)$ and $v = \psi(x)$. In this case z is a composite function of one variable x

$$z = f(x, \varphi(x), \psi(x))$$

The derivative of that function $\dfrac{dz}{dx}$ we obtain using (1.22). As the derivative $\dfrac{dx}{dx} = 1$ and u and v are the functions of one variable, then

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial u}\frac{du}{dx} + \frac{\partial z}{\partial v}\frac{dv}{dx}. \tag{1.24}$$

The derivative in (1.24) is called the *total derivative.*

Example 2. Find $\dfrac{dz}{dx}$ for $z = x^2 + \sqrt{y}$ and $y = x^2 + 1$.

Here z is the function of two variables x and y, where y is the function of the variable x. In this case the formula (1.24) gives

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 2x + \frac{1}{2\sqrt{y}} \cdot 2x = x\left(2 + \frac{1}{\sqrt{y}}\right) = x\left(2 + \frac{1}{\sqrt{x^2+1}}\right).$$

1.9 Higher order partial derivatives

As we have seen in many examples, the partial derivatives of the function $z = f(x, y)$ $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are in general functions of two variables again. Thus, it is possible to differentiate both of them with respect to x and y.

Definition 1. The partial derivative with respect to x of the partial derivative $\dfrac{\partial z}{\partial x}$ is called the *second order partial derivative with respect to x* and denoted $\dfrac{\partial^2 z}{\partial x^2}$ (to be read *de-squared-zed de-ex-squared*), that means

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

Definition 2. The partial derivative with respect to y of the partial derivative $\dfrac{\partial z}{\partial x}$ is called the *second order partial derivative with respect to x* and y and denoted $\dfrac{\partial^2 z}{\partial x \partial y}$ (to be read *de-squared-zed de-ex-de-y*). By this definition

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

Definition 3. The partial derivative with respect to x of the partial derivative $\dfrac{\partial z}{\partial y}$ is called the *second order partial derivative with respect to y* and x and denoted $\dfrac{\partial^2 z}{\partial y \partial x}$, that is

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

Definition 4. The partial derivative with respect to y of the partial derivative $\dfrac{\partial z}{\partial y}$ is called the *second order partial derivative with respect to y*

and denoted $\dfrac{\partial^2 z}{\partial y^2}$, i.e

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

The second and third second order partial derivatives are often called *mixed partial derivatives* since we are taking derivatives with respect to more than one variable.

The second order partial derivatives are denoted also z''_{xx}, z''_{xy}, z''_{yx} and z''_{yy} or $f''_{xx}(x,y)$, $f''_{xy}(x,y)$, $f''_{yx}(x,y)$ and $f''_{yy}(x,y)$.

The second order partial derivatives are the functions of two variables x and y again. Hence, all four second order partial derivatives can be differentiated with respect to x and y. So we define eight third order partial derivatives

$$\frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right), \quad \frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right)$$

$$\frac{\partial^3 z}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x \partial y}\right), \quad \frac{\partial^3 z}{\partial x \partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x \partial y}\right)$$

$$\frac{\partial^3 z}{\partial y \partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y \partial x}\right), \quad \frac{\partial^3 z}{\partial y \partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y \partial x}\right)$$

$$\frac{\partial^3 z}{\partial y^2 \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right), \quad \frac{\partial^3 z}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y^2}\right)$$

Example 1. Find all second order partial derivatives for $z = \arctan\dfrac{x}{y}$.

In Example 2 of subsection 6.5 we have found

$$\frac{\partial z}{\partial x} = \frac{y}{x^2+y^2} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{x}{x^2+y^2}$$

We find

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = y\frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right) = y\left(-\frac{2x}{(x^2+y^2)^2}\right) = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{x^2+y^2-y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right) = -\frac{x^2+y^2-x\cdot 2x}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{x}{x^2+y^2}\right) = -x\frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right) = -x\left(-\frac{2y}{(x^2+y^2)^2}\right) = \frac{2xy}{(x^2+y^2)^2}$$

These results suggest a question, are the mixed second order partial derivatives

$$\frac{\partial^2 z}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 z}{\partial y \partial x}$$

equal. The next theorem says that if the function is smooth enough this will always be the case.

Theorem. If the function $z = f(x, y)$ and its partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ are continuous at the point P and on some neighborhood of this point, then at the point P

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

This theorem says that if the partial derivatives to be evaluated are continuous, then the result of repeated differentiation is independent of the order in which it is performed.

Therefore, if the partial derivatives involved are continuous, the also holds

$$\frac{\partial^4 z}{\partial x \partial y \partial x \partial y} = \frac{\partial^4 z}{\partial x^2 \partial y^2} = \frac{\partial^4 z}{\partial y^2 \partial x^2}$$

Analogous theorem is valid also for the functions of three etc. variables.

Example 2. Find the third order partial derivatives $\frac{\partial^3 w}{\partial x \partial y \partial z}$ and $\frac{\partial^3 w}{\partial z \partial x \partial y}$ for the function of three variables $w = e^x \sin(yz)$.

First we find

$$\frac{\partial w}{\partial x} = e^x \sin(yz)$$

second

$$\frac{\partial^2 w}{\partial x \partial y} = e^x \cos(yz) \cdot z = ze^x \cos(yz)$$

and third

$$\frac{\partial^3 w}{\partial x \partial y \partial z} = e^x \cos(yz) + z(-e^x \sin(yz)) \cdot y = e^x[\cos(yz) - yz \sin(yz)]$$

To find the second third order partial derivative, we find

$$\frac{\partial w}{\partial z} = ye^x \cos(yz)$$

next

$$\frac{\partial^2 w}{\partial z \partial x} = ye^x \cos(yz)$$

and finally

$$\frac{\partial^3 w}{\partial z \partial x \partial y} = e^x \cos(yz) - ye^x \sin(yz) \cdot z = e^x[\cos(yz) - yz\sin(yz)]$$

1.10 Directional derivative

Up to now for the function of two variables $z = f(x,y)$ we've only looked at the two partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. Recall that these derivatives represent the rate of change of f as we vary x (holding y fixed) and as we vary y (holding x fixed) respectively. We now need to discuss how to find the rate of change of $f(x,y)$ if we allow both x and y to change simultaneously. In other words how to find the rate of change of $f(x,y)$ in the direction of vector $\overrightarrow{s} = (\Delta x, \Delta y)$.

The goal is to obtain the formula to compute the derivative of the function $z = f(x,y)$ at the point $P(x,y)$ in the direction of the vector $\overrightarrow{s} = (\Delta x, \Delta y)$.

Assume that the function $z = f(x,y)$ and its partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are continuous at P and in some neighborhood of this point.

Denote the length of the vector $\overrightarrow{s}$ by $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$. By the (1.8) the total increment of the function has the form

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and ε_2 are infinitesimals as $\Delta s \to 0$. Dividing the last equality by the length of the vector $\overrightarrow{s}$ gives

$$\frac{\Delta z}{\Delta s} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta s} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta s} + \varepsilon_1 \frac{\Delta x}{\Delta s} + \varepsilon_2 \frac{\Delta y}{\Delta s}$$

The ratios $\dfrac{\Delta x}{\Delta s}$ and $\dfrac{\Delta y}{\Delta s}$ are the coordinates of the unit vector $\overrightarrow{s^0}$ in direction of the vector $\overrightarrow{s}$. Denoting by α and β the angles that $\overrightarrow{s}$ forms with the coordinate axes, it's obvious that

$$\frac{\Delta x}{\Delta s} = \cos\alpha \quad \text{and} \quad \frac{\Delta y}{\Delta s} = \cos\beta$$

Therefore, these ratios, i.e. the coordinates of the unit vector in direction of the vector $\overrightarrow{s}$ are called the *directional cosines* of that vector.

Definition. The limit
$$\lim_{\Delta s \to 0} \frac{\Delta z}{\Delta s}$$
is called the derivative of z at the point P in the direction of the vector $\overrightarrow{s}$ and denoted $\dfrac{\partial z}{\partial \overrightarrow{s}}$. Since
$$\lim_{\Delta s \to 0} \left(\varepsilon_1 \frac{\Delta x}{\Delta s} + \varepsilon_2 \frac{\Delta y}{\Delta s} \right) = 0$$
we have the formula to compute the directional derivative
$$\frac{\partial z}{\partial \overrightarrow{s}} = \frac{\partial z}{\partial x} \cos\alpha + \frac{\partial z}{\partial y} \cos\beta \tag{1.25}$$

Example 1. Find the derivatives of the function $z = x^2 + y^2$ at the point $P(1; 1)$ in directions of vectors $\overrightarrow{s_1} = (1; 1)$ and $\overrightarrow{s_2} = (1; -1)$.

First we evaluate the partial derivatives of z at P
$$\frac{\partial z}{\partial x} = 2x \bigg|_P = 2$$
and
$$\frac{\partial z}{\partial y} = 2y \bigg|_P = 2$$
The length of the vector $\overrightarrow{s_1}$ is $\Delta s_1 = \sqrt{2}$, the directional cosines are $\cos\alpha = \dfrac{1}{\sqrt{2}}$ and $\cos\beta = \dfrac{1}{\sqrt{2}}$. Hence,
$$\frac{\partial z}{\partial \overrightarrow{s_1}} = 2 \cdot \frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}} = 2\sqrt{2}$$
The length of the vector $\overrightarrow{s_2}$ is $\Delta s_2 = \sqrt{2}$, the directional cosines are $\cos\alpha = \dfrac{1}{\sqrt{2}}$ and $\cos\beta = -\dfrac{1}{\sqrt{2}}$. Thus,
$$\frac{\partial z}{\partial \overrightarrow{s_2}} = 2 \cdot \frac{1}{\sqrt{2}} - 2 \cdot \frac{1}{\sqrt{2}} = 0$$

Starting from the same point in the xy plane and moving in different directions, we get the different results. Thus, the directional derivative has no meaning without specifying the direction. The directional derivative gives us the instantaneous rate of change of the given function of two variables at a certain point in the pre-scribed direction.

Partial derivatives with respect to x and y are special cases of the directional derivative. If the given vector $\overrightarrow{s}$ points in direction of x-axis then $\alpha = 0$, $\beta = \dfrac{\pi}{2}$, $\cos\alpha = 1$ and $\cos\beta = 0$. Hence,

$$\frac{\partial z}{\partial \overrightarrow{s}} = \frac{\partial z}{\partial x}$$

If the given vector $\overrightarrow{s}$ points in direction of y-axis then $\alpha = \dfrac{\pi}{2}$, $\beta = 0$, $\cos\alpha = 0$ and $\cos\beta = 1$. It follows

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial y}$$

Thus, the directional derivative in the direction of x axis is the partial derivative with respect to x and the directional derivative in the direction of y-axis is the partial derivative with respect to y.

The directional derivative of the function of three variables $w = f(x, y, z)$ at the point $P(x, y, z)$ in the direction of the vector $\overrightarrow{s} = (\Delta x, \Delta y, \Delta z)$ can be found by the similar formula. Let α, β and γ denote the angles between the vector $\overrightarrow{s}$ and x-axis, y-axis and z-axis respectively. Then the directional cosines of the vector $\overrightarrow{s}$ are $\cos\alpha$, $\cos\beta$ and $\cos\gamma$. The directional derivative is computed by the formula

$$\frac{\partial w}{\partial \overrightarrow{s}} = \frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\cos\beta + \frac{\partial w}{\partial z}\cos\gamma \tag{1.26}$$

Example 2. Find the directional derivative of the function $w = xy + xz + yz$ at the point $P(1; 1; 2)$ in the direction of the vector that makes with the coordinate axes the angles $60°$, $60°$ and $45°$ respectively.

Find the partial derivatives at the point P

$$\frac{\partial w}{\partial x} = y + z\Big|_P = 3, \quad \frac{\partial w}{\partial y} = x + z\Big|_P = 3$$

and

$$\frac{\partial w}{\partial z} = x + y\Big|_P = 2$$

and the directional cosines

$$\overrightarrow{s^{\circ}} = (\cos 60^{\circ}; \cos 60^{\circ}; \cos 45^{\circ}) = \left(\frac{1}{2}; \frac{1}{2}; \frac{\sqrt{2}}{2}\right).$$

By the formula (3.35) we obtain

$$\frac{\partial w}{\partial \overrightarrow{s}} = 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} + 2 \cdot \frac{\sqrt{2}}{2} = 3 + \sqrt{2}$$

1.11 Gradient

The function of two variables $z = f(x, y)$ associates to any point $P(x, y)$ in the domain of that function D one value of the dependent variable z or a scalar. To any point in the domain of the function there is related a scalar. Hence, the function of two variables creates a *scalar field* in the plane.

The function of two variables $w = f(x, y, z)$ associates to any point $P(x, y, z)$ in its domain V a scalar, i.e creates a scalar field in the domain V. Examples used in physics include the temperature distribution throughout space, the pressure distribution in a fluid or in a gas. Scalar fields are contrasted with other physical quantities such as vector fields, which associate a vector to every point of a region.

Definition 1.

$$\operatorname{grad} z = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \tag{1.27}$$

is called the *gradient* of the scalar field $z = f(x, y)$.

Definition 2. The vector

$$\operatorname{grad} w = \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right) \tag{1.28}$$

is called the gradient of the scalar field $w = f(x, y, z)$.

In the first case there is defined a vector field in the plane and in the second case a vector field in the space. These are called the *gradient field*.

If $\overrightarrow{s^{\circ}} = (\cos\alpha, \cos\beta)$ denotes the unit vector in the direction of the vector $\overrightarrow{s}$, the formula (1.25) can be written as the scalar product of the gradient and the unit vector $\overrightarrow{s^{\circ}}$

$$\frac{\partial z}{\partial \overrightarrow{s}} = \operatorname{grad} z \cdot \overrightarrow{s^{\circ}}$$

Since $\overrightarrow{s^{\circ}} = \dfrac{\overrightarrow{s}}{\Delta s}$, then

$$\frac{\partial z}{\partial \overrightarrow{s}} = \operatorname{grad} z \cdot \frac{\overrightarrow{s}}{\Delta s} = |gradz| \frac{\operatorname{grad} z \cdot \overrightarrow{s}}{|\operatorname{grad} z| \Delta s}$$

where $|\operatorname{grad} z|$ is the length of the gradient vector. Denoting by φ the angle between the gradient and the vector $\overrightarrow{s}$ we obtain

$$\cos\varphi = \frac{\operatorname{grad} z \cdot \overrightarrow{s}}{|\operatorname{grad} z|\Delta s}$$

and

$$\frac{\partial z}{\partial \overrightarrow{s}} = |\operatorname{grad} z| \cos\varphi. \tag{1.29}$$

Now we formulate this result as a theorem.

Theorem 1. The directional derivative of the function $z = f(x, y)$ equals to the projection of the gradient vector onto the direction of vector $\overrightarrow{s}$.

Two important conclusions of this theorem.

Conclusion 1. The directional derivative in direction perpendicular to the gradient equals to zero.

This conclusion is obvious because in our case $\varphi = \dfrac{\pi}{2}$ and $\dfrac{\partial z}{\partial \overrightarrow{s}} = 0$.

Conclusion 2. The directional derivative has the greatest value in the direction of the gradient and equals to the length of the gradient.

It's enough to recall that the cosine function obtains its greatest value 1 if $\varphi = 0$. Thus, the direction of fastest change for a function is given by the gradient vector at that point.

Example 1. Find the greatest rate of growth of the function $z = x^2 + y^2$ at the point $P(1; 1)$.

The directional derivative gives the instantaneous rate of change at the given point. The greatest instantaneous rate of change equals to the length of the gradient. We find the gradient vector at the point P

$$\operatorname{grad} z = (2x, 2y)\Big|_P = (2; 2)$$

and its length $|\operatorname{grad} z| = 2\sqrt{2}$.

This result is the same as the result in Example 1 of the previous subsection, where we have found the directional derivative in direction of the vector $\overrightarrow{s_1}$. This is natural because the vector $\overrightarrow{s_1} = (1; 1)$ and the gradient have the same directions.

Theorem 2. The gradient is perpendicular to the tangent of level curve.

Proof. The projection of the level curve of the surface $z = f(x, y)$ onto xy-plane is $f(x, y) = c$. This is an implicit function of one variable and the graph is a curve in xy-plane. The slope of the tangent line of this curve is $\dfrac{dy}{dx} = -\dfrac{f'_x}{f'_y}$. Hence, the direction vector of the tangent line is

$$\overrightarrow{s} = \left(1; -\frac{f'_x}{f'_y}\right) = \frac{1}{f'_y}(f'_y, -f'_x)$$

The scalar product of the gradient vector and the direction vector of the tangent line

$$\operatorname{grad} z \cdot \overrightarrow{s} = f'_x f'_y - f'_y f'_x = 0$$

which means that these two vectors are perpendicular.

Now the Conclusion 1 gives us.

Conclusion 3. The derivative in the direction of the tangent line of the level curve equals to zero.

In Example 1 of the previous subsection the vector $\overrightarrow{s_2}$ has the same direction as the tangent line of the level curve. Thus, by Conclusion 3 it is natural that the derivative in the direction of this vector equals to zero.

Definition 3. A vector field $\overrightarrow{F} = (X(x,y), Y(x,y))$ is called a *conservative vector field* if there exists a scalar field $z = f(x,y)$ such that $\overrightarrow{F} = \operatorname{grad} z$. If $\overrightarrow{F}$ is a conservative vector field then the function $f(x,y)$ is called a *potential function* for $\overrightarrow{F}$.

All this definition is saying is that a vector field is conservative if it is also a gradient vector field for some scalar field.

Example 2. The vector field $\overrightarrow{F} = (2xy; x^2)$ is conservative because there exists the scalar field $z = x^2y$ such that $\operatorname{grad} z = \overrightarrow{F}$ and x^2y is the potential function for $\overrightarrow{F}$.

1.12 Divergence and curl

The gradient vector field is just one example of vector fields. More generally, a vector field $\overrightarrow{F} = (X(x,y,z); Y(x,y,z); Z(x,y,z))$ is an assignment of a vector to each point (x,y,z) in a subset of space. Vector fields are often used to model, for example, the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point or the speed and direction of a moving fluid throughout space.

Definition 1. The scalar

$$\operatorname{div} \overrightarrow{F} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \tag{1.30}$$

is called the *divergence* of the vector field $\overrightarrow{F}$ at the point $P(x,y,z)$.

Definition 2. The vector

$$\operatorname{curl} \overrightarrow{F} = \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}; \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}; \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) \tag{1.31}$$

is called the *curl* (or rotor) of the vector field $\overrightarrow{F}$ at the point $P(x,y,z)$.

Example 1. Find the divergence and curl of the vector field $\overrightarrow{F} = \left(xyz; x^2+z^2; \dfrac{xy}{z}\right)$.

In this example $X = xyz$, $Y = x^2+z^2$, $Z = \dfrac{xy}{z}$, thus, $\dfrac{\partial X}{\partial x} = yz$, $\dfrac{\partial Y}{\partial y} = 0$ and $\dfrac{\partial Z}{\partial z} = -\dfrac{xy}{z^2}$. Hence, the divergence

$$\operatorname{div} \overrightarrow{F} = yz - \frac{xy}{z^2}$$

The components of the curl vector

$$\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} = \frac{x}{z} - 2z$$

$$\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} = xy - \frac{y}{z}$$

and

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = 2x - xz$$

Consequently,

$$\operatorname{curl} \overrightarrow{F} = \left(\frac{x}{z} - 2z; xy - \frac{y}{z}; 2x - xz\right)$$

If the vector field represents the velocity of a moving flow in space, then the divergence of a vector field $\overrightarrow{F}$ at point $P(x, y, z)$ represents a measure of the rate at which the flow diverges (spreads away) from P. That is, $\operatorname{div} \overrightarrow{F}\big|_P$ is the limit of the flow per unit volume out of the infinitesimal sphere centered at P. The curl represents the rotation of a flow, i.e. $\operatorname{curl} \overrightarrow{F}\big|_P$ measures the extent to which the vector field $\overrightarrow{F}$ rotates around P.

Suppose that $\overrightarrow{F}$ is the velocity field in a flowing fluid. Then the curl $\overrightarrow{F}$ represents the tendency of particles at the point (x, y, z) to rotate about the axis that points in direction of curl $\overrightarrow{F}$. The length of curl vector represents the velocity of that rotation.

If $\operatorname{curl} \overrightarrow{F} = \overrightarrow{0}$, the vector field $\overrightarrow{F}$ is called *irrotational.*

In field theory there is used a formal vector.

Definition 3. The vector

$$\nabla = \left(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right)$$

is called *Hamilton nabla vector* or *Hamilton nabla operator.*

The coordinates of this vector are not numbers but some operators. The first coordinate means that we find the partial derivative with respect to x for some function etc.

If we treat this vector as an usual vector, we can write for the scalar field $w = f(x, y, z)$

$$\nabla w = \left(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right) w = \left(\frac{\partial w}{\partial x}; \frac{\partial w}{\partial y}; \frac{\partial w}{\partial z}\right) = \operatorname{grad} w$$

Here we have the formal scalar multiplication of ∇ and w. The order of factors is important. The quantities on which ∇ acts must appear to the right of ∇.

The scalar product of ∇ and the vector field $\overrightarrow{F} = (X(x, y, z); Y(x, y, z); Z(x, y, z))$ is

$$\nabla \cdot \overrightarrow{F} = \left(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right) \cdot (X; Y; Z) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = \operatorname{div} \overrightarrow{F}$$

The vector product of ∇ and the vector field $\overrightarrow{F} = (X(x, y, z); Y(x, y, z); Z(x, y, z))$ is

$$\nabla \times \overrightarrow{F} = \left(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right) \times (X; Y; Z) = \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}; \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}; \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) = \operatorname{curl} \overrightarrow{F}$$

Hence, using the nabla operator, we can write

$$\operatorname{grad} w = \nabla w$$

$$\operatorname{div} \overrightarrow{F} = \nabla \cdot \overrightarrow{F}$$

$$\operatorname{curl} \overrightarrow{F} = \nabla \times \overrightarrow{F}$$

Definition 4. The scalar product of nabla vector by itself $\nabla^2 = \nabla \cdot \nabla$ is called Laplacian operator and denoted

$$\triangle = \nabla^2$$

The scalar product of nabla vector by itself is not a real quantity

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

but applying this operator to some function, we obtain at every point of the space a scalar.

Example 2. Find the Laplacian operator for the function $w = e^x \sin(yz)$.

First we find the first-order partial derivatives

$$\frac{\partial w}{\partial x} = e^x \sin(yz)$$

$$\frac{\partial w}{\partial y} = ze^x \cos(yz)$$
$$\frac{\partial w}{\partial z} = ye^x \cos(yz)$$

and next

$$\begin{aligned} \triangle w &= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \\ &= e^x \sin(yz) - z^2 e^x \sin(yz) - y^2 e^x \sin(yz) \\ &= e^x \sin(yz)(1 - z^2 - y^2) = w(1 - z^2 - y^2) \end{aligned}$$

Finally we prove some equalities that hold for the scalar field $w = f(x, y, z)$ and vector field $\overrightarrow{F} = (X; Y; Z)$.

Corollary 1. $\operatorname{div} \operatorname{grad} w = \triangle w$

Proof We write

$$\operatorname{div} \operatorname{grad} w = \nabla \cdot \operatorname{grad} w = \left(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial w}{\partial x}; \frac{\partial w}{\partial y}; \frac{\partial w}{\partial z}\right) = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

1.13 Local extrema of function of two variables

The theory of maxima and minima for the functions of two variables is similar to the theory for one variable.

Definition 1. It is said that the function of two variables $f(x, y)$ has a local maximum at the point $P_1(x_1, y_1)$, if there exists a neighborhood of this point $U_\varepsilon(x_1, y_1)$ such that for any $P(x, y) \in U_\varepsilon(x_1, y_1)$

$$f(x, y) < f(x_1, y_1)$$

Definition 2. It is said that the function of two variables $f(x, y)$ has a local minimum at the point $P_2(x_2, y_2)$, if there exists a neighborhood of this point $U_\varepsilon(x_2, y_2)$ such that for any $P(x, y) \in U_\varepsilon(x_2, y_2)$

$$f(x, y) > f(x_2, y_2)$$

Local extremum is either a local maximum or a local minimum.

Example 1. By Definition 2 the function $z = x^2 + y^2$ has the local minimum at the point $P_0(0; 0)$ because $f(0; 0) = 0$ and for any point $P(x, y)$ different of P_0 there holds $f(x, y) = x^2 + y^2 > 0$.

Example 2. The function $z = x^2 - y^2$ has no local extremum at the point $P_0(0; 0)$. We have $f(0; 0) = 0$ and any neighborhood $U_\varepsilon(0; 0)$ contains the points of x-axis and y-axis. At the points on x-axis $y = 0$ and $z = x^2 > 0$, at the points of y-axis $x = 0$ and $z = -y^2 < 0$.

If the function of two variables has local extremum at the point $P_0(x_0, y_0)$ then the intersection curve of surface (the graph of the function of two variables) and the plain $y = y_0$ has local extremum at x_0. Hence, the function of one variable $z = f(x, y_0)$ has local extremum at x_0. It follows that at the point P_0 either $\dfrac{\partial z}{\partial x} = 0$ or does not exist.

As well, the intersection curve of surface and the plain $x = x_0$ has local extremum at y_0. The function of one variable $z = f(x_0, y)$ has local extremum at y_0. Then at the point P_0 either $\dfrac{\partial z}{\partial y} = 0$ or does not exist.

Definition 3. The points, where $\dfrac{\partial z}{\partial x} = 0$ or does not exist and $\dfrac{\partial z}{\partial y} = 0$ or does not exist, are called the *critical points* of the function of two variables.

Now we can formulate the theorem.

Theorem 1. (Necessary condition for existence of local extremum). If the function $z = f(x, y)$ has local extremum at the point P_0, then P_0 is the critical point of this function.

This theorem says that the function of two variables has a local extremum only at the critical point of this function. But the condition given in this theorem is not sufficient for the function to have a local extremum. For instance the point $O(0; 0)$ is the critical point of the function $z = x^2 - y^2$ because the partial derivatives $\dfrac{\partial z}{\partial x} = 2x$ and $\dfrac{\partial z}{\partial y} = 2y$ both equal to zero at this point, but as we know by Example 2, this function has no local maximum and local minimum at $O(0; 0)$.

Because of this theorem we know that if we have all the critical points of a function then we also have every possible local extremum for the function. The fact tells us that all local extrema must be at the critical points so we know that if the function does have local extrema then they must be in the set of all the critical points. However, it will be completely possible that at least at one of the critical points the function hasn't a local extremum.

So the question is how to determine whether the function of two variables has a local extremum at the critical point or not and if it has, is at that point a local maximum or a local minimum.

In the following we consider only the critical points where both partial derivatives equal to zero, i.e. the system of equations

$$\begin{cases} \dfrac{\partial z}{\partial x} = 0 \\ \dfrac{\partial z}{\partial y} = 0 \end{cases} \tag{1.32}$$

The solutions of this system of equations are called the *stationary points* of the function $z = f(x, y)$. Every stationary point is also a critical point of the function of two variables but not vice versa. There exist the critical points that are not the stationary points. For instance, for the function $z = \sqrt{x^2 + y^2}$ the partial derivatives

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$$

and

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

are never simultaneously zero, however they both don't exist at $O(0;0)$. Therefore, $O(0;0)$ is a critical point and a possible extremum. The graph of $z = \sqrt{x^2 + y^2}$ is a cone opening upwards with vertex at the origin. Therefore, at $O(0;0)$ this function has a local minimum at $O(0;0)$.

We find the sufficient conditions for existence of the local extremum at the stationary points. Let P_0 be a stationary point of the function $z = f(x, y)$. Evaluate the second order partial derivatives at P_0 and denote

$$A = \left.\frac{\partial^2 z}{\partial x^2}\right|_{P_0} \quad B = \left.\frac{\partial^2 z}{\partial x \partial y}\right|_{P_0} \quad \text{and} \quad C = \left.\frac{\partial^2 z}{\partial y^2}\right|_{P_0}$$

Theorem 2 (sufficient conditions for existence of a local extremum). Let P_0 be a stationary point of the function $z = f(x, y)$.

1. If $AC - B^2 > 0$ and $A < 0$ then the function $z = f(x, y)$ has a local maximum at P_0.

2. If $AC - B^2 > 0$ and $A > 0$ then the function $z = f(x, y)$ has a local minimum at P_0.

3. If $AC - B^2 < 0$ then the function $z = f(x, y)$ has no local extremum at P_0.

Definition 4. If $AC - B^2 < 0$ then the stationary point P_0 is called the *saddle point* of the function $z = f(x, y)$.

We obtain the stationary point $P_0(0;0)$ of the function $z = x^2 + y^2$ as the solution of the system of equations (1.32)

$$\begin{cases} 2x = 0 \\ 2y = 0 \end{cases}$$

We find

$$A = \frac{\partial^2 z}{\partial x^2} = 2$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = 0$$

and

$$C = \frac{\partial^2 z}{\partial y^2} = 2$$

Hence, $AC - B^2 = 4 > 0$ and $A > 0$. Consequently, by Theorem 2 the function $z = x^2 + y^2$ has at stationary point $P_0(0;0)$ a local minimum.

We obtain the stationary point $P_0(0;0)$ of the function $z = x^2 - y^2$ as the solution of the system of equations (1.32)

$$\begin{cases} 2x = 0 \\ -2y = 0 \end{cases}$$

We find

$$A = \frac{\partial^2 z}{\partial x^2} = 2$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = 0$$

and

$$C = \frac{\partial^2 z}{\partial y^2} = -2$$

Thus, $AC - B^2 = -4 < 0$. Consequently, by Theorem 2 the function $z = x^2 - y^2$ has't a local extremum at the stationary point $P_0(0;0)$. In other words: the point $P_0(0;0)$ is the saddle point of the function $z = x^2 - y^2$.

Example 3. Find the local extrema of the function $f(x,y) = 4 + x^3 + y^3 - 3xy$.

The first order partial derivatives are

$$\frac{\partial f}{\partial x} = 3x^2 - 3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 3y^2 - 3x$$

To find the stationary points we solve the system of equations (1.32)

$$\begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases}$$

or

$$\begin{cases} x^2 - y = 0 \\ y^2 - x = 0 \end{cases}$$

The first equation gives $y = x^2$. Substituting this into second equation gives $x^4 - x = 0$ or $x(x^3 - 1) = 0$, whose solutions are $x_1 = 0$ and $x_2 = 1$. Since $y = x^2$, we have two stationary points $P_1(0;0)$ and $P_2(1;1)$. Next we find the second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = 6x \quad \frac{\partial^2 f}{\partial x \partial y} = -3 \;\text{ and }\; \frac{\partial^2 f}{\partial y^2} = 6y$$

Since at the first stationary point $P_1(0;0)$ the values $A = 0$, $B = -3$ and $C = 0$ and

$$AC - B^2 = 0 \cdot 0 - (-3)^2 = -9$$

the point $P_1(0;0)$ is the saddle point of the given function.

At the second stationary point $P_2(1;1)$ the values $A = 6$, $B = -3$ and $C = 6$ and

$$AC - B^2 = 6 \cdot 6 - (-3)^2 = 27 > 0$$

As well $A = 6 > 0$ and by Theorem 2 the given function has a local minimum at the point $P_2(1;1)$ and this local minimum equals to

$$z_{min} = 4 + 1^3 + 1^3 - 3 \cdot 1 \cdot 1 = 3$$

Remark If in Theorem 2 $AC - B^2 = 0$ then anything is possible. More advanced methods are required to classify the stationary point properly.

2 Multiple integrals

We have finished our discussion of partial derivatives of functions of more than one variable and we move on to integrals of functions of two or three variables.

2.1 Definition and properties of double integral

Consider the function of two variables $f(x, y)$ defined in the bounded region D. Divide the region D into randomly selected n subregions

$$\Delta s_1,\ \Delta s_2,\ \ldots,\ \Delta s_k,\ \ldots, \Delta s_n$$

where Δs_k, $1 \le k \le n$, denotes the kth subregion or the area of this subregion.

Next we choose a random point in every subregion $P_k(\xi_k, \eta_k) \in \Delta s_k$ and multiply the value of the function at the point chosen by the area of the subregion $f(P_k)\Delta s_k$. If we assume that $f(P_k) \ge 0$ then this product equals to the volume of the right prism with the area of base Δs_k and the height $f(P_k)$.

The sum

$$\sum_{k=1}^{n} f(P_k)\Delta s_k$$

is called the *integral sum* of the function $f(x, y)$ over the region D. The geometric meaning is the sum of the volumes of the right prisms, provided $f(x, y) \ge 0$ in the region D.

The maximal distance between the points of the subregion Δs_k is called the *diameter* of this subregion

$$\operatorname{diam} \Delta s_k = \max_{P,Q \in \Delta s_k} |\overrightarrow{PQ}|$$

We have divided the region into subregions randomly. Every subregion has its own diameter. The greatest diameter of subregions we denote by λ, i.e.

$$\lambda = \max_{1 \le k \le n} \operatorname{diam} \Delta s_k$$

Definition 1. If there exists the limit

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} f(P_k)\Delta s_k$$

and this limit does not depend of the choice of subregions of D and the choice of the points P_k in subregions, then this limit is called the *double integral* of the function of two variables $f(x, y)$ over the region D and denoted

$$\iint_D f(x, y)dxdy$$

According to this definition

$$\iint_D f(x, y)dxdy = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(P_k)\Delta s_k \tag{2.1}$$

If $f(x, y) \geq 0$ in the region D then the double integral can be interpreted as the volume of the cylinder between the surface $z = f(x, y)$ and D.

There holds the following theorem.

Theorem 1. If $f(x, y)$ is continuous in the bounded region D then

$$\iint_D f(x, y)dxdy$$

always exists.

The proof will be omitted. This theorem tells us that for the continuous function $f(x, y)$ the limit

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} f(P_k)\Delta s_k$$

exists and does not depend of the choice of subregions of D and the choice of the points P_k in subregions.

All of the following properties are really just extensions of properties of single integrals.

Property 1. The double integral of the sum of two functions equals to the sum of double integrals of these functions

$$\iint_D [f(x, y) + g(x, y)]dxdy = \iint_D f(x, y)dxdy + \iint_D g(x, y)dxdy$$

provided all three double integrals exist.

Proof. We use the properties of the sum and the limit. By Definition 1

$$\iint\limits_D [f(x,y)+g(x,y)]dxdy = \lim_{\lambda\to 0}\sum_{k=1}^{n}[f(P_k)+g(P_k)]\Delta s_k$$

$$= \lim_{\lambda\to 0}\left[\sum_{k=1}^{n} f(P_k)\Delta s_k + \sum_{k=1}^{n} g(P_k)\Delta s_k\right]$$

$$= \lim_{\lambda\to 0}\sum_{k=1}^{n} f(P_k)\Delta s_k + \lim_{\lambda\to 0}\sum_{k=1}^{n} g(P_k)\Delta s_k.$$

By Definition 1 the first limit equals to $\iint\limits_D f(x,y)dxdy$ and the second limit equals to $\iint\limits_D g(x,y)dxdy$.

Property 2. If c is a constant then

$$\iint\limits_D cf(x,y)dxdy = c\iint\limits_D f(x,y)dxdy$$

i.e. the constant factor can be carried outside the sign of the double integral.

The proof is similar to the proof of Property 1.

Property 3. The double integral of the difference of two functions equals to the difference of double integrals of these functions

$$\iint\limits_D [f(x,y)-g(x,y)]dxdy = \iint\limits_D f(x,y)dxdy - \iint\limits_D g(x,y)dxdy.$$

Property 3 is the conclusion of the properties 1 and 2 because

$$f(x,y)-g(x,y) = f(x,y)+(-1)g(x,y)$$

Property 4. If $D = D_1 \cup D_2$ and the regions D_1 and D_2 have not common interior points then

$$\iint\limits_D f(x,y)dxdy = \iint\limits_{D_1} f(x,y)dxdy + \iint\limits_{D_2} f(x,y)dxdy$$

Proof. In the definition of the double integral the limit doesn't depend on the division of the region D. Therefore, starting the random division of the region D we first divide D into D_1 and D_2. Further random division of the region D creates the random divisions into subregions of the regions D_1 and D_2. The integral sum we split into two addends

$$\sum_{k=1}^{n} f(P_k)\Delta s_k = \sum_{D_1} f(P_k)\Delta s_k + \sum_{D_2} f(P_k)\Delta s_k \tag{2.2}$$

where the first addend contains the products, having as one factor the area of the subregions of the region D_1 and the second addend contains the products, having as one factor the area of the subregions of the region D_2, The first sum on the right side of this equality is the integral sum of the function $f(x, y)$ over the region D_1 and the second over the region D_2.

If λ denotes the greatest diameter of the subregions of the region D then $\lambda \to 0$ yields that the greatest diameter of the subregions of D_1 and D_2 approach to zero. We get the assertion of our property if we find the limits of both sides of the equality (2.2) as $\lambda \to 0$.

2.2 Iterated integral. Evaluation of double integral

In the previous subsection we have defined the double integral. However, just like with the definition of a definite integral the definition is very difficult to use in practice and so we need to start looking into how we actually compute double integrals. In this subsection we assume that the bounded region D is closed. There are two types of regions that we need to look at.

The region D is called *regular with respect to y axis* if any straight line parallel to y axis passing the interior points of the region cuts the boundary at two points.

The regular with respect to y axis region can be described by two pairs of inequalities $a \le x \le b$ and $\varphi_1(x) \le y \le \varphi_2(x)$. This notation is a way of saying we are going to use all the points, (x, y), in which both of the coordinates satisfy the given inequalities.

Let the function $f(x, y)$ be defined in the region D. The *iterated integral* of the function $f(x, y)$ over this region is defined as follows

$$\int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \right) dx$$

To compute the iterated integral we integrate first with respect to y by holding x constant as if this were a definite integral. This is called *inner integral* and the result of this integration is the function of one variable x

$$\Phi(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy$$

Second we multiply the function obtained by dx and compute the *outer integral*

$$\int_a^b \Phi(x)dx$$

which is another definite integral. So, to compute the iterated integral we have to compute two definite integrals. First we integrate with respect to inner variable y and second with respect to outer variable x. To avoid the parenthesis we shall further write the iterated integral as

$$\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \tag{2.3}$$

Example 1. Compute the iterated integral

$$\int_0^1 dx \int_0^{x^2} (x^2+y)dy$$

The region of integration D is described by the inequalities $0 \le x \le 1$ and $0 \le y \le x^2$.

First we compute the inner integral by treating x as constant

$$\Phi(x) = \int_0^{x^2} (x^2+y)dy = \left(x^2y + \frac{y^2}{2}\right)\Big|_0^{x^2} = x^4 + \frac{x^4}{2} = \frac{3x^4}{2}$$

and then the outer integral

$$\int_0^1 \frac{3x^4}{2} dx = \frac{3}{2}\frac{x^5}{5}\bigg|_0^1 = \frac{3}{10}$$

The region D is called *regular with respect to x axis* if any straight line parallel to x axis passing the interior points of the region cuts the boundary at two points.

The regular with respect to x axis region can be described by two pairs of inequalities $c \le y \le d$ and $\psi_1(y) \le x \le \psi_2(y)$.

The iterated integral over the region D regular with respect to x axis is defined as

$$\int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx. \tag{2.4}$$

To compute this iterated integral we have to find two definite integrals again. First we integrate with respect to inner variable x by holding y constant. The result is a function of one variable y

$$\Psi(y) = \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx$$

Second we integrate with respect to outer variable y

$$\int_c^d \Psi(y)dy$$

In the iterated integral (2.3) the variable y is the inner variable and x is the outer variable, in the iterated integral (2.4) the situation is vice versa. The conversion of the iterated integral from one order of integration to other order of integration is called the *change (or the reverse) of the order of integration.*

Example 2. Change the order of integration in the iterated integral

$$\int_0^1 dx \int_0^{x^2} f(x,y)dy$$

In the iterated integral given the inner variable is y and the outer variable x. After changing the order of integration the outer variable has to be y and the inner variable x. Note that the limits of the outer variable are always constants. The limits of the inner variable are in general (but not always) the functions of the outer variable.

Choosing y as the outer variable, this changes between 0 and 1 that is $0 \le y \le 1$. Solving the equation $y = x^2$ for x we get $x = \pm\sqrt{y}$. The domain in this example is bounded by the right branch of the parabola $x = \sqrt{y}$, thus, the variable x in this region is determined by $\sqrt{y} \le x \le 1$. Changing the order of integration, we obtain the iterated integral

$$\int_0^1 dy \int_{\sqrt{y}}^1 f(x, y)dx$$

Let us evaluate the iterated integral of Example 1 again, using the reversed order of integration, i.e. compute

$$\int_0^1 dy \int_{\sqrt{y}}^1 (x^2 + y)dx$$

Here we integrate first with respect to x

$$\int_{\sqrt{y}}^1 (x^2 + y)dx = \left(\frac{x^3}{3} + yx\right)\Bigg|_{\sqrt{y}}^1 = \frac{1}{3} + y - \frac{y\sqrt{y}}{3} - y\sqrt{y} = \frac{1}{3} + y - \frac{4}{3}y^{\frac{3}{2}}$$

Next we integrate with respect to y

$$\int_0^1 \left(\frac{1}{3} + y - \frac{4}{3}y^{\frac{3}{2}}\right) dy = \left[\frac{y}{3} + \frac{y^2}{2} - \frac{8y^2\sqrt{y}}{15}\right]\Bigg|_0^1 = \frac{3}{10}$$

The result is, as expected, equal to the result obtained in Example 1.

Example 4. Sometimes we need to change the order of integration to get a tractable integral. For example, if we try to evaluate

$$\int_0^1 dx \int_x^1 e^{y^2} dy$$

directly, we shall run into trouble because there is no antiderivative of e^{y^2}, so we get stuck trying to compute the integral with respect to y. But, if we change the order of integration, then we can integrate with respect to x first, which is doable. And, it turns out that the integral with respect to y also becomes possible after we finish integrating with respect to x. If we choose

the reversed order for integration then the region of integration is described by inequalities $0 \le y \le 1$ and $0 \le x \le y$, thus

$$\int_0^1 dx \int_x^1 e^{y^2} dy = \int_0^1 dy \int_0^y e^{y^2} dx$$

Evaluating the inner integral with respect to x we treat y as constant, i.e. e^{y^2} is a constant factor and

$$\int_0^y e^{y^2} dx = e^{y^2} \cdot x \Big|_0^y = ye^{y^2}$$

Now it is possible to find the outer integral, using the differential $d(y^2) = 2ydy$

$$\int_0^1 ye^{y^2} dy = \frac{1}{2} \int_0^1 e^{y^2} d(y^2) = \frac{1}{2} e^{y^2} \Big|_0^1 = \frac{e-1}{2}$$

Example 4. Change the order of integration in the iterated integral

$$\int_0^3 dy \int_y^{6-y} f(x,y)dx. \tag{2.5}$$

The region of integration is described by inequalities $0 \le y \le 3$ and $y \le x \le 6-y$. We sketch in the Figure the lines $y = 0$, $y = 3$, $x = y$ and $x = 6 - y$. Obviously in this region $0 \le x \le 6$ and $0 \le y \le \varphi(x)$, where

$$\varphi(x) = \begin{cases} x, & \text{if } 0 \le x \le 3 \\ 6 - x, & \text{if } 3 \le x \le 6 \end{cases}$$

By the additivity property of the definite integral

$$\begin{aligned}\int_0^6 dx \int_0^{\varphi(x)} f(x,y)dy &= \int_0^3 dx \int_0^{\varphi(x)} f(x,y)dy + \int_3^6 dx \int_0^{\varphi(x)} f(x,y)dy \\ &= \int_0^3 dx \int_0^{x} f(x,y)dy + \int_3^6 dx \int_0^{6-x} f(x,y)dy\end{aligned}$$

Dividing the region D by the line $x = 3$ we obtain two regular regions D_1 and D_2. The region D_1 is determined by the inequalities $0 \le x \le 3$ and $0 \le y \le x$ and D_2 by inequalities $3 \le x \le 6$ and $0 \le y \le 6 - x$. Consequently, if we want to change the order of integration in the iterated integral (2.5), we have to divide the region of integration by the line $x = 3$ into two regions D_1 and D_2 and determine the limits for both regions.

Why have we paid so much attention to iterated integrals? The answer is given by the following theorem.

Theorem. If the function $f(x,y)$ is continuous in the closed regular region D then

$$\iint_D f(x,y)dxdy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \tag{2.6}$$

This theorem states that the iterated integral equals to the double integral and so we don't need the term iterated integral any more. It's just a mean to compute the double integral and usually we shall say instead of iterated integral double integral.

If the region D is regular with respect to x axis then we compute the double integral by the formula

$$\iint_D f(x,y)dxdy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx. \tag{2.7}$$

If the region is regular with respect to either of the coordinate axes then we can choose one of these formulas to compute the double integral. Sometimes it is unimportant, which order of integration we choose. Sometimes one order of integration leads to less computational work that another order of integration. Sometimes (recall Example 4) the computation of double integral is possible for one order of integration but it is impossible for another order.

Example 5. Compute the double integral $\iint\limits_D (x+y)dxdy$ if D is the region bounded by the line $x+y=2$ and parabola $y=x^2$.

To sketch the region we find the intersection points of the parabola and line solving the system of equations

$$\begin{cases} y = x^2 \\ x+y=2 \end{cases}$$

The second equation gives $y = 2-x$. Substituting y to the first equation gives the quadratic equation $x^2+x-2=0$, whose roots are $x_1=-2$ and $x_2=1$.

First we determine the limits of integration $-2 \le x \le 1$ and $x^2 \le y \le 2-x$. By the formula (2.6)

$$\iint\limits_D (x+y)dxdy = \int\limits_{-2}^{1} dx \int\limits_{x^2}^{2-x} (x+y)dy$$

Next we compute the inner integral (we integrate with respect to y, so x is constant)

$$\begin{aligned} &\int\limits_{x^2}^{2-x} (x+y)dy = \left(xy + \frac{y^2}{2}\right)\Bigg|_{x^2}^{2-x} \\ = \ &x(2-x) + \frac{(2-x)^2}{2} - x^3 - \frac{x^4}{2} = 2 - \frac{x^2}{2} - x^3 - \frac{x^4}{2} \end{aligned}$$

and finally the outer integral

$$\begin{aligned} &\int\limits_{-2}^{1} \left(2 - \frac{x^2}{2} - x^3 - \frac{x^4}{2}\right) dx = \left(2x - \frac{x^3}{6} - \frac{x^4}{4} - \frac{x^5}{10}\right)\Bigg|_{-2}^{1} \\ = \ &2 - \frac{1}{6} - \frac{1}{4} - \frac{1}{10} - \left(-4 + \frac{4}{3} - 4 + \frac{16}{5}\right) = 4,95. \end{aligned}$$

2.3 Change of variables in double integral

Often the reason for changing variables is to get us an integral that we can do with the new variables. Another reason for changing variables is to convert the region into a nicer region to work with. The following example gives an idea why the change of variable can be useful.

Example 1. Compute the double integral

$$\iint\limits_D (2x - 3y - 4)^2 dxdy$$

where D is the region bounded by the lines $x+y = -1$, $x+y = 3$, $3y-2x = 6$ and $2x - 3y = 12$.

Notice that the first two lines are parallel and the third and fourth lines are parallel, i.e. the region D in this example is a parallelogram. The intersection point of the first and third line is $A\left(-\frac{9}{5}; \frac{4}{5}\right)$, the intersection point of the first and fourth line is $B\left(\frac{9}{5}; -\frac{14}{5}\right)$, the intersection point of the second and fourth line is $C\left(\frac{21}{5}; -\frac{6}{5}\right)$ and the intersection point of the second and third line is $D\left(\frac{3}{5}; \frac{12}{5}\right)$

To compute this double integral by the formula (2.6), we have to divide the region with two vertical lines into three subregions, compute this double integral over these three subregions and add the results. The integration demands quite a lot of technical work, which can be avoided if we use the change of variables.

Change the variables x and y by the variables u and v by the equations

$$\begin{cases} x = \varphi(u, v) \\ y = \psi(u, v) \end{cases} \tag{2.8}$$

We assume that $x = \varphi(u, v)$ and $y = \psi(u, v)$ are one-valued continuous functions in the respective region of the uv plane and they have continuous partial derivatives with respect to both variables in that region. In addition we assume that the system of equations (2.8) can be uniquely solved for the variables u and v. Then to any point of the region D in xy plane there is related one point of the region D' in uv plane and vice versa.

Recall while changing the variable in definite integral we had to express the differential of the old variable via the differential of the new variable. In double integral we change two variables and this relationship is accomplished by the functional determinant called *jacobian*

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \tag{2.9}$$

Now the formula of the change of variables in the double integral is

$$\iint\limits_{D} f(x,y)dxdy = \iint\limits_{D'} f(\varphi(u,v),\psi(u,v))|J|dudv \tag{2.10}$$

Let us return to Example 1 of this subsection at let's change the variable setting

$$\begin{cases} u = x + y \\ v = 2x - 3y \end{cases} \tag{2.11}$$

This way the parallelogram in xy plane converts to the rectangle in uv plane determined by the inequalities $-1 \leq u \leq 3$ and $-6 \leq v \leq 12$.The integrand converts to $(2x-3y-4)^2 = (v-4)^2$. To compute the jacobian (2.9) we solve the system of equations (2.11) for the variables x and y

$$\begin{cases} x = \dfrac{3}{5}u + \dfrac{1}{5}v \\ \\ y = \dfrac{2}{5}u - \dfrac{1}{5}v \end{cases}$$

and find the partial derivatives

$$\frac{\partial x}{\partial u} = \frac{3}{5}, \quad \frac{\partial y}{\partial u} = \frac{2}{5}$$
$$\frac{\partial x}{\partial v} = \frac{1}{5}, \quad \frac{\partial y}{\partial v} = -\frac{1}{5}$$

Thus, the jacobian

$$J = \begin{vmatrix} \dfrac{3}{5} & \dfrac{2}{5} \\ \\ \dfrac{1}{5} & -\dfrac{1}{5} \end{vmatrix} = -\frac{3}{25} - \frac{2}{25} = -\frac{1}{5}$$

and by the formula of change of variable (2.10) we find

$$\iint\limits_{D} (2x-3y-4)^2 dxdy = \iint\limits_{D'} (v-4)^2 \frac{1}{5} dudv = \frac{1}{5}\int\limits_{-1}^{3} du \int\limits_{-6}^{12} (v-4)^2 dv = 403\frac{1}{5}$$

2.4 Double integral in polar coordinates

If we substitute the Cartesian coordinates x and y by the polar coordinates φ and ρ, we use the formulas

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases} \tag{2.12}$$

Recall that φ denotes the polar angle and ρ the polar radius.

To the constant polar angles there correspond the straight lines passing the origin in xy plane and to the constant polar radius there correspond the circles centered at the origin in xy plane. Therefore, first of all we use the change of variable (2.12) if the region of integration is a disk or the part of disk.

To use the formula (2.10) we find $f(x, y) = f(\rho\cos\varphi, \rho\sin\varphi)$ and the jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial\varphi} & \frac{\partial y}{\partial\varphi} \\ \frac{\partial x}{\partial\rho} & \frac{\partial y}{\partial\rho} \end{vmatrix} = \begin{vmatrix} -\rho\sin\varphi & \rho\cos\varphi \\ \cos\varphi & \sin\varphi \end{vmatrix} = -\rho$$

Since ρ is the polar radius, which is non-negative, we have $|J| = \rho$.

Suppose the region D in xy plane converts to the region Δ in $\varphi\rho$ plane. Then the general formula (2.10) gives us the formula to convert the double integral into polar coordinates

$$\iint\limits_D f(x, y)dxdy = \iint\limits_\Delta f(\rho\cos\varphi, \rho\sin\varphi)\rho d\varphi d\rho \tag{2.13}$$

Example 1. Convert to polar coordinates the double integral

$$\iint\limits_D f(x, y)dxdy$$

if the region of integration D is the disk $x^2 + y^2 \le 4y$.

The region D is bounded by the circle $x^2 + y^2 = 4y$. Converting this equation, we have $x^2+y^2-4y = 0$, i.e. $x^2+y^2-4y+4 = 4$ or $x^2+(y-2)^2 = 4$. This is the equation of the circle with radius 2 centered at the point $(0; 2)$.

The x axis is the tangent line of this circle, hence, the polar angle changes from 0 to π, i.e. $0 \le \varphi \le \pi$. The least value of the polar radius is 0 for any polar angle. The greatest value of the polar radius depends on the polar angle. To get this dependence, we convert by (2.12) the equation of the circle $x^2 + y^2 = 4y$ into polar coordinates

$$\rho^2\cos^2\varphi + \rho^2\sin^2\varphi = 4\rho\sin\varphi$$

which yields

$$\rho = 4\sin\varphi$$

Thus in the disk given, the polar radius satisfies the inequalities $0 \leq \rho \leq 4\sin\varphi$.

By the formula (2.13) we get

$$\iint\limits_{D} f(x,y)dxdy = \iint\limits_{\Delta} f(\rho\cos\varphi, \rho\sin\varphi)\rho d\varphi d\rho$$

where Δ is determined by the inequalities $0 \leq \varphi \leq \pi$ and $0 \leq \rho \leq 4\sin\varphi$. Using the iterated integral, we obtain

$$\iint\limits_{D} f(x,y)dxdy = \int\limits_{0}^{\pi} d\varphi \int\limits_{0}^{4\sin\varphi} f(\rho\cos\varphi, \rho\sin\varphi)\rho d\rho$$

Example 2. Using the polar coordinates, compute the double integral

$$\iint\limits_{D} \frac{dxdy}{x^2+y^2+1}$$

if the region of integration D is bounded by $y = 0$ and $y = \sqrt{1-x^2}$.

Since $x^2+y^2+1 = \rho^2\cos^2\varphi + \rho^2\sin^2\varphi + 1 = \rho^2+1$, then

$$\iint\limits_{D} \frac{dxdy}{x^2+y^2+1} = \iint\limits_{\Delta} \frac{\rho d\varphi d\rho}{\rho^2+1}$$

The region D is bounded by x axis and the upper half of the circle $x^2+y^2 = 1$.

The region D in Cartesian coordinates converts to the region Δ in polar coordinates, determined by the inequalities $0 \leq \varphi \leq \pi$ and $0 \leq \rho \leq 1$. Thus,

$$\iint\limits_{\Delta} \frac{\rho d\varphi d\rho}{\rho^2+1} = \int\limits_{0}^{\pi} d\varphi \int\limits_{0}^{1} \frac{\rho d\rho}{\rho^2+1}$$

First we find the inside integral. Since the differential of the denominator $d(\rho^2+1) = 2\rho d\rho$, then

$$\int\limits_{0}^{1} \frac{\rho d\rho}{\rho^2+1} = \frac{1}{2}\int\limits_{0}^{1} \frac{d(\rho^2+1)}{\rho^2+1} = \frac{1}{2}\ln(\rho^2+1)\Big|_{0}^{1} = \frac{1}{2}\ln 2$$

Now, the outside integral

$$\int\limits_{0}^{\pi} \frac{1}{2}\ln 2 d\varphi = \frac{1}{2}\ln 2\int\limits_{0}^{\pi} d\varphi = \frac{\pi}{2}\ln 2$$

2.5 Computation of areas and volumes by double integrals

While defining the double integral we got the geometrical meaning of this. Assuming that $f(x,y) \geq 0$ in the region D, the double integral

$$\iint\limits_D f(x,y)dxdy$$

is the volume of the solid enclosed by the region D in the xy plane, the graph of the function $z = f(x,y)$ and the cylinder surface, whose generatrix is parallel to z axis. This idea can be extended to more general regions.

Suppose that in the region D for two functions $f(x,y)$ and $g(x,y)$ there holds $f(x,y) \geq g(x,y)$. The property of the double integral gives

$$\iint\limits_D [f(x,y) - g(x,y)]dxdy = \iint\limits_D f(x,y)dxdy - \iint\limits_D g(x,y)dxdy$$

Geometrically both integrals on the right mean the volumes of the solids. The first integral equals to the volume of the solid that lies below the surface $z = f(x,y)$ and above the region D in the xy plane. The second integral is the volume of the solid that lies between the surface $z = g(x,y)$ and the region D.

Thus, the volume is computed by the formula

$$V = \iint\limits_D [f(x,y) - g(x,y)]dxdy \tag{2.14}$$

This is the volume of the solid enclosed by the surface $z = f(x,y)$ from the top, by surface $z = g(x,y)$ from the bottom and the cylinder surface, whose directrix is the boundary of the region D and generatrix is parallel to z axis.

Example 1. Compute the volume of the solid enclosed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

The pyramid is bounded by the plane $z = 1 - x - y$ from the top and by xy-plane $z = 0$ from the bottom. Using the formula (2.14) $f(x, y) = 1-x-y$ and $g(x, y) = 0$. The volume of the pyramid is

$$V = \iint\limits_D (1 - x - y)dxdy$$

The region D is determined by inequalities $0 \le x \le 1$ and $0 \le y \le 1 - x$, thus,

$$V = \int\limits_0^1 dx \int\limits_0^{1-x} (1 - x - y)dy$$

First we compute the inside integral

$$\int\limits_0^{1-x} (1-x-y)dy = -\int\limits_0^{1-x} (1-x-y)d(1-x-y) = -\frac{(1-x-y)^2}{2}\bigg|_0^{1-x} = \frac{(1-x)^2}{2}$$

and second the outside integral

$$V = \int\limits_0^1 \frac{(1-x)^2}{2}dx = -\int\limits_0^1 \frac{(1-x)^2}{2}d(1-x) = -\frac{(1-x)^3}{6}\bigg|_0^1 = \frac{1}{6}$$

If the height of the solid $f(x, y) = 1$ at any point of the region D, then the volume of this solid $V = S_D \cdot 1$, where S_D is the area of the bottom (the region D). So, in this case the area of the bottom and the volume of the solid are numerically equal. Substituting the function $f(x, y) = 1$ into the formula

$$V = \iint\limits_D f(x, y)dxdy$$

we get the formula to compute the area of the plane region D

$$S_D = \iint\limits_D dxdy \tag{2.15}$$

Example 2. Compute the area of the region bounded by the lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

Converting the equation of the lemniscate into polar coordinates, we obtain $\rho = a\sqrt{\cos 2\varphi}$. Hence, $-\frac{\pi}{2} \leq 2\varphi \leq \frac{\pi}{2}$, which yields $-\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}$ or $\frac{3\pi}{4} \leq \varphi \leq \frac{5\pi}{4}$.

The lemniscate is symmetrical with respect to the x axis and y axis. Therefore we compute the area of the region D bounded by the lemniscate in the first quadrant of the coordinate plane and multiply the result by 4. Converting the the formula (2.15) to polar coordinates gives

$$S = 4\iint\limits_{D} dxdy = 4\iint\limits_{\Delta} \rho d\varphi d\rho$$

The region of integration Δ is determined by the inequalities $0 \leq \varphi \leq \frac{\pi}{4}$ and $0 \leq \rho \leq a\sqrt{\cos 2\varphi}$. Hence,

$$S = 4\int\limits_{0}^{\frac{\pi}{4}} d\varphi \int\limits_{0}^{a\sqrt{\cos 2\varphi}} \rho d\rho$$

Computing the inside integral gives

$$\int\limits_{0}^{a\sqrt{\cos 2\varphi}} \rho d\rho = \left.\frac{\rho^2}{2}\right|_{0}^{a\sqrt{\cos 2\varphi}} = \frac{a^2}{2}\cos 2\varphi$$

and the outside integral

$$S = 4\int\limits_{0}^{\frac{\pi}{4}} \frac{a^2}{2}\cos 2\varphi d\varphi = a^2\int\limits_{0}^{\frac{\pi}{4}} \cos 2\varphi d(2\varphi) = a^2 \sin 2\varphi\Big|_{0}^{\frac{\pi}{4}} = a^2$$

2.6 Definition and properties of triple integral

We used a double integral to integrate over a two-dimensional region and so it's natural that we'll use a triple integral to integrate over a three-dimensional region. Suppose the function of three variables $f(x, y, z)$ is defined in the three dimensional region V. Choose the whatever partition of the region V into n subregions

$$\Delta v_1,\ \Delta v_2,\ \ldots,\ \Delta v_k,\ \ldots, \Delta v_n$$

where Δv_k denotes the kth subregion, as well the volume of this subregion.

For each subregion, we pick a random point $P_k(\xi_k, \eta_k, \zeta_k) \in \Delta v_k$ to represent that subregion and find the product of the value of the function at that point and the volume of subregion $f(P_k)\Delta v_k$. Adding all these products, we obtain the sum

$$\sum_{k=0}^{n} f(P_k)\Delta v_k, \tag{2.16}$$

which is called the *integral sum* of the function $f(x, y, z)$ over the region V.

Let

$$\operatorname{diam} \Delta v_k = \max_{P,Q \in \Delta v_k} |\overrightarrow{PQ}|$$

be the diameter of the subregion Δv_k and λ the greatest diameter of the subregions, i.e.

$$\lambda = \max_{0 \le k \le n} \operatorname{diam} \Delta v_k$$

Definition. If there exists the limit

$$\lim_{\lambda \to 0} \sum_{k=0}^{n} f(P_k)\Delta v_k$$

and this limit doesn't depend on the partition of the region V and the choice of the points P_k in the subregions, then this limit is called the triple integral of the function $f(x, y, z)$ over the region V and denoted

$$\iiint\limits_V f(x, y, z)dxdydz$$

Thus, by this definition

$$\iiint\limits_V f(x, y, z)dxdydz = \lim_{\lambda \to 0} \sum_{k=0}^{n} f(P_k)\Delta v_k \tag{2.17}$$

If the function $f(x, y, z)$ is continuous in the closed region V, then the triple integral (2.17) always exists.

The properties of the triple integral are quite similar to the properties of the double integral.

Property 1.

$$\iiint\limits_V [f(x, y, z) \pm g(x, y, z)]dxdydz = \iiint\limits_V f(x, y, z)dxdydz \pm \iiint\limits_V g(x, y, z)dxdydz$$

Property 2. If c is a constant then

$$\iiint\limits_V cf(x,y,z)dxdydz = c\iiint\limits_V f(x,y,z)dxdydz$$

Property 3. If $V = V_1 \cup V_2$ and the regions V_1 and V_2 have no common interior point then

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_{V_1} f(x,y,z)dxdydz + \iiint\limits_{V_2} f(x,y,z)dxdydz$$

Let $f(x,y,z) \geq 0$ be the density of a three-dimensional solid V at the point (x,y,z) inside the solid. By picking a point P_k to represent the subregion Δv_k we treat the density $f(P_k)$ constant in the subregion Δv_k and the product $f(P_k)\Delta v_k$ is the approximate mass of the subregion Δv_k. The approximate mass because we have substituted the variable density $f(x,y,z)$ by the constant density $f(P_k)$.

The integral sum is the sum of the approximate masses of the subregions, i.e. the approximate mass of the region V. The limiting process $\lambda \to 0$ means that all diameters of the subregions are infinitesimals. The density at the point P_k represents the density of the subregion Δv_k with the greater accuracy and the integral sum will approach to the total mass of the region V.

Therefore, if the function $f(x,y,z) \geq 0$ is the density of a three-dimensional solid V then the triple integral equals to the mass of the solid V

$$m = \iiint\limits_V f(x,y,z)dxdydz$$

If the region V has the uniform density 1, then the mass and volume are numerically equal, i.e. if $f(x,y,z) \equiv 1$, then the volume of the region V is computable by the formula

$$V = \iiint\limits_V dxdydz \tag{2.18}$$

An example how to use this formula we have later.

2.7 Evaluation of triple integral

The region V in the space is called *regular in direction of z axis* if there are satisfied three conditions.

1. Any line parallel to the z axis passing the interior point of this region cuts the boundary surface at two points.

2. The projection of the region onto xy plane is a regular plain region.

3. Cutting the region by the plane parallel to some coordinate plane creates two regions satisfying the conditions 1. and 2.

If those conditions are fulfilled, then the region V is determined by inequalities $a \le x \le b$, $\varphi_1(x) \le y \le \varphi_2(x)$ and $\psi_1(x,y) \le z \le \psi_2(x,y)$. We can define the iterated integral

$$\int\limits_a^b dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} dy \int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz$$

To compute this iterated integral we have to compute three definite integrals. First we integrate with respect to the variable z holding x and y constant

$$\Psi(x,y) = \int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz$$

We call this inside integral and the result is a function of two variables $\Psi(x,y)$. Next we integrate with respect to the intermediate variable y holding x constant

$$\Phi(x) = \int\limits_{\varphi_1(x)}^{\varphi_2(x)} \Psi(x,y)dy$$

The result is a function of the one variable $\Phi(x)$. Finally we compute the outside integral

$$\int\limits_a^b \Phi(x)dx$$

Notice that the limits of the outside variable a and b are always constants. The limits of the intermediate variable $\varphi_1(x)$ and $\varphi_2(x)$ depend in general on the outside variable. The limits of the inside variable $\psi_1(x,y)$ and $\psi_2(x,y)$ depend in general on the outside variable and on the intermediate variable.

Example 1. Compute the iterated integral $\int\limits_0^1 dx \int\limits_0^x dy \int\limits_0^{xy} (x+y)dz$

First we integrate with respect to the inner variable z. Since $x+y$ is constant, then

$$\int_0^{xy} (x+y)dz = (x+y)\cdot z\Big|_0^{xy} = x^2y + xy^2$$

This result we integrate with respect to intermediate variable y

$$\int_0^{x} (x^2y+xy^2)dy = x^2\cdot\frac{y^2}{2}\Big|_0^x + x\cdot\frac{y^3}{3}\Big|_0^x = \frac{x^4}{2}+\frac{x^4}{3} = \frac{5x^4}{6}$$

and finally with respect to x

$$\int_0^1 \frac{5x^4}{6}dx = \frac{5}{6}\cdot\frac{x^5}{5}\Big|_0^1 = \frac{1}{6}$$

Since we have assumed that the projection of the region V onto xy plane is a regular plane region, then the region V can be determined by inequalities $c\le y\le d$, $\varphi_1(y)\le x\le\varphi_2(y)$ and $\psi_1(x,y)\le z\le\psi_2(x,y)$ and the iterated integral can be defined as

$$\int_c^d dy \int_{\varphi_1(y)}^{\varphi_2(y)} dx \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz$$

Just like we have defined the regular region in direction of z axis, we can define the regular region in direction of x axis and the regular region in direction of y axis. In the first case it is possible to define the iterated integrals

$$\int_a^b dy \int_{\varphi_1(y)}^{\varphi_2(y)} dz \int_{\psi_1(y,z)}^{\psi_2(y,z)} f(x,y,z)dx$$

or

$$\int_a^b dz \int_{\varphi_1(z)}^{\varphi_2(z)} dy \int_{\psi_1(x,y)}^{\psi_2(y,z)} f(x,y,z)dx$$

and in the second case

$$\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dz \int_{\psi_1(x,z)}^{\psi_2(x,z)} f(x,y,z)dy$$

or

$$\int\limits_{a}^{b} dz \int\limits_{\varphi_1(z)}^{\varphi_2(z)} dx \int\limits_{\psi_1(x,z)}^{\psi_2(x,z)} f(x,y,z)dy$$

So, if the region V is regular in direction of all coordinate axes, six orders of integration are possible. The conversion of the iterated integral for one order of integration to the iterated integral for another order of integration is called the change of the order of integration.

The iterated integral has the most simple limits, if the region of integration is a rectangular box defined by $a \le x \le b$, $c \le y \le d$ and $p \le z \le q$. All the faces of that box are parallel to one of three coordinate planes.

If we choose x the outer variable, y the intermediate variable and z the inner variable, we compute

$$\int\limits_{a}^{b} dx \int\limits_{c}^{d} dy \int\limits_{p}^{q} f(x,y,z)dy$$

and, of course, five more orders of integration are possible.

The iterated integral is the appropriate tool to compute the triple integral.

Theorem. If the function $f(x,y,z)$ is continuous in the closed regular region V, then

$$\iiint\limits_{V} f(x,y,z)dxdydz = \int\limits_{a}^{b} dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} dy \int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz \tag{2.19}$$

Example 2. Compute the triple integral

$$\iiint\limits_{V} xyzdxdydz$$

if the region V is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x+y+z = 1$.

First three planes are the coordinate planes. The fourth plane passes three points $(1;0;0)$, $(0;1;0)$ and $(0;0;1)$. The intersection line of this plane and xy plane $z = 0$ is $x + y = 1$.

The projection of the region of integration onto the xy plane is the triangle, which is determined by equalities $0 \le x \le 1$ and $0 \le y \le 1-x$. Since the region of integration is bounded by the plane $z = 0$ on the bottom and by $z = 1-x-y$ on the top, the region of integration is determined by $0 \le x \le 1$, $0 \le y \le 1-x$ and $0 \le z \le 1-x-y$. By the formula (2.19)

$$\iiint\limits_V xyzdxdydz = \int\limits_0^1 dx \int\limits_0^{1-x} dy \int\limits_0^{1-x-y} xyzdz$$

First we compute the inside integral

$$\int\limits_0^{1-x-y} xyzdz = xy\frac{z^2}{2}\Bigg|_0^{1-x-y} = xy\frac{(1-x-y)^2}{2}$$

Next we integrate with respect to y

$$\int\limits_0^{1-x} xy\frac{(1-x-y)^2}{2}dy = \frac{x}{2}\int\limits_0^{1-x} y[(1-x)^2 - 2(1-x)y + y^2]dy$$

$$= \frac{x}{2}\left[(1-x)^2\frac{y^2}{2} - 2(1-x)\frac{y^3}{3} + \frac{y^4}{4}\right]\Bigg|_0^{1-x}$$

$$= \frac{x}{2}\left[\frac{(1-x)^4}{2} - \frac{2(1-x)^4}{3} + \frac{(1-x)^4}{4}\right] = \frac{x(1-x)^4}{24}$$

and finally

$$\frac{1}{24}\int\limits_0^1 x(1-x)^4dx = -\frac{1}{24}\int\limits_0^1 (-x)(1-x)^4dx$$

$$= -\frac{1}{24}\int\limits_0^1 (1-x-1)(1-x)^4dx = \frac{1}{24}\int\limits_0^1 [(1-x)^5 - (1-x)^4]d(1-x)$$

$$= \frac{1}{24}\left[\frac{(1-x)^6}{6} - \frac{(1-x)^5}{5}\right]\Bigg|_0^1 = \frac{1}{24}(-\frac{1}{6} + \frac{1}{5}) = \frac{1}{720}$$

2.8 Change of variable in triple integral

Changing variables in triple integrals is nearly identical to changing variables in double integrals. We are going to change the variables in the triple integral

$$\iiint\limits_V f(x,y,z)dxdydz$$

over the region V in the xyz space. We use the transformation

$$\begin{cases} x = \varphi(u,v,w) \\ y = \psi(u,v,w) \\ z = \chi(u,v,w) \end{cases} \tag{2.20}$$

to transform the region V into the new region V' in the uvw space. We assume that the functions x, y and z of the variables u, v and w are one-valued and the system of equations (2.20) has unique solution for u, v and w. Then to any point in the region V' there is related one point in the region V and vice versa. In addition we assume that the functions (2.20) are continuous and they have continuous partial derivatives with respect to all three variables in the region V'.

The jacobian of this change of variables is the determinant

$$J = \begin{vmatrix} x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \\ x'_w & y'_w & z'_w \end{vmatrix} \tag{2.21}$$

and we can transform the triple integral over the region V into the triple integral over the region V' by the formula

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_{V'} f(\varphi(u,v,w),\psi(u,v,w),\chi(u,v,w))|J|dudvdw \tag{2.22}$$

2.9 Triple integral in cylindrical coordinates

The cylindrical coordinates are really nothing more than an extension of polar coordinates into three dimensions leaving the z coordinate unchanged. For the given point $P(x,y,z)$ in the xyz space we denote P' the projection of this point onto xy plane. Denote by ρ the distance of P' from the origin and by φ the angle between the segment $P'O$ and x axis. Those φ and ρ are exactly the same as the polar coordinates in the two-dimensional case.

Definition. The cylindrical coordinates of the point P are called φ, ρ and z.

Since φ and ρ have in the xy plane the same meaning as the polar coordinates then the conversion formulas from the Cartesian coordinates into cylindrical coordinates are

$$\begin{cases} x = \rho \cos \varphi \\ y = \rho \sin \varphi \\ z = z \end{cases} \tag{2.23}$$

Find the jacobian of this change of variables. By the formula (2.21) we get

$$J = \begin{vmatrix} x'_\varphi & y'_\varphi & z'_\varphi \\ x'_\rho & y'_\rho & z'_\rho \\ x'_z & y'_z & z'_z \end{vmatrix}$$

The variable z does not depend on φ and ρ, hence, $z'_\varphi = 0$ and $z'_\rho = 0$. The variables x and y does not depend on z, i.e. $x'_z = 0$ and $y'_z = 0$. Consequently,

$$J = \begin{vmatrix} -\rho \sin \varphi & \rho \cos \varphi & 0 \\ \cos \varphi & \sin \varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding this determinant by the last column gives

$$J = \begin{vmatrix} -\rho \sin \varphi & \rho \cos \varphi \\ \cos \varphi & \sin \varphi \end{vmatrix} = -\rho \sin^2 \varphi - \rho \cos^2 \varphi = -\rho$$

Since ρ is a distance $|J| = \rho$.

Let V' be the region in cylindrical coordinates, which corresponds to the region V in Cartesian coordinates. By the general formula for change of variables in the triple integral (2.22) we obtain the formula to convert the triple integral in Cartesian coordinates into the triple integral in cylindrical coordinates

$$\iiint\limits_V f(x, y, z) dx dy dz = \iiint\limits_{V'} f(\rho \cos \varphi, \rho \sin \varphi, z) \rho d\varphi d\rho dz \tag{2.24}$$

Supposing that the region V' in cylindrical coordinates is given by the inequalities $\alpha \le \varphi \le \beta$, $\rho_1(\varphi) \le \rho \le \rho_2(\varphi)$ and $z_1(\varphi, \rho) \le z \le z_2(\varphi, \rho)$, we can write by the formula (2.19)

$$\iiint\limits_V f(x, y, z) dx dy dz = \int\limits_\alpha^b d\varphi \int\limits_{\rho_1(\varphi)}^{\rho_2(\varphi)} d\rho \int\limits_{z_1(\varphi,\rho)}^{z_2(\varphi,\rho)} f(\rho \cos \varphi, \rho \sin \varphi, z) \rho dz \tag{2.25}$$

Example 1. Convert $\int\limits_{-1}^{1} dy \int\limits_{0}^{\sqrt{1-y^2}} dx \int\limits_{x^2+y^2}^{\sqrt{x^2+y^2}} f(x,y,z)dz$ into an integral in cylindrical coordinates.

The ranges of the variables in Cartesian coordinates from this iterated integral are

$$\begin{gathered} -1 \leq y \leq 1 \\ 0 \leq x \leq \sqrt{1-y^2} \\ x^2+y^2 \leq z \leq \sqrt{x^2+y^2} \end{gathered}$$

The first two inequalities define the projection D of this region onto xy-plane, which is the half of the disk of radius 1 centered at the origin. The third equality determines that the region of integration is bounded by the paraboloid of rotation $z = x^2 + y^2$ on the bottom and by the cone $z = \sqrt{x^2+y^2}$ on the top.

In cylindrical coordinates the equation on the paraboloid of rotation converts to $z = \rho^2$ and the equation of the cone to $z = \rho$. So, the ranges for the region of integration in cylindrical coordinates are,

$$\begin{gathered} -\tfrac{\pi}{2} \leq \varphi \leq \tfrac{\pi}{2} \\ 0 \leq \rho \leq 1 \\ \rho^2 \leq z \leq \rho \end{gathered}$$

Now, by the formula (2.25) we write

$$\int\limits_{-1}^{1} dy \int\limits_{0}^{\sqrt{1-y^2}} dx \int\limits_{x^2+y^2}^{\sqrt{x^2+y^2}} f(x,y,z)dz = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_{0}^{1} d\rho \int\limits_{\rho^2}^{\rho} f(\rho\cos\rho, \rho\sin\varphi, z)\rho dz$$

Notice that the limits of integration are simpler in the cylindrical coordinates.

Example 2. Using the cylindrical coordinate, compute the triple integral

$$\int\limits_{0}^{2} dx \int\limits_{0}^{\sqrt{2x-x^2}} dy \int\limits_{0}^{a} z\sqrt{x^2+y^2}dz$$

In Cartesian coordinates the region of integration is defined by the inequalities $0 \leq x \leq 2$, $0 \leq y \leq \sqrt{2x-x^2}$ and $0 \leq z \leq a$, i.e. bounded by the planes

$x=0$, $x=2$, $y=0$, $z=0$ and $z=a$ and by the cylinder $y=\sqrt{2x-x^2}$. The generatrix of the cylinder is parallel to the z axis and the projection onto xy plane is the half circle $y=\sqrt{2x-x^2}$. This is the upper half of the circle $y^2=2x-x^2$ or $x^2-2x+y^2=0$, i.e.

$$(x-1)^2+y^2=1$$

which is the circle of radius 1 centered at $(1;0)$.

Convert the integral given into the integral in cylindrical coordinates. The range of the angle φ in the projection of this region onto xy plane is $0\le\varphi\le\dfrac{\pi}{2}$. Converting the equation of the cylinder $x^2+y^2=2x$ into cylindrical coordinates gives $\rho^2\cos^2\varphi+\rho^2\sin^2\varphi=2\rho\cos\varphi$ or $\rho=2\cos\varphi$. Hence, the range for ρ is $0\le\rho\le 2\cos\varphi$. We didn't convert the third coordinate z, thus, $0\le z\le a$.

Converting the integrand into cylindrical coordinates gives

$$z\sqrt{\rho^2\cos^2\varphi+\rho^2\sin^2\varphi}=z\rho$$

Now, by the formula by (2.25)

$$\int_0^2 dx\int_0^{\sqrt{2x-x^2}}dy\int_0^a z\sqrt{x^2+y^2}dz=\int_0^{\frac{\pi}{2}}d\varphi\int_0^{2\cos\varphi}d\rho\int_0^a z\rho\cdot\rho dz$$

The integration with respect to z gives

$$\int_0^a z\rho^2 dz=\rho^2\frac{z^2}{2}\bigg|_0^a=\frac{a^2\rho^2}{2}$$

the integration with respect to ρ gives

$$\frac{a^2}{2}\int_0^{2\cos\varphi}\rho^2 d\varrho=\frac{a^2}{2}\frac{\rho^3}{3}\bigg|_0^{2\cos\varphi}=\frac{4a^2\cos^3\varphi}{3}$$

Finally, integrating with respect to φ, we get

$$\frac{4a^2}{3}\int_0^{\frac{\pi}{2}}\cos^3\varphi d\varphi=\frac{4a^2}{3}\int_0^{\frac{\pi}{2}}(1-\sin^2\varphi)d(\sin\varphi)=\frac{4a^2}{3}\left(\sin\varphi-\frac{\sin^3\varphi}{3}\right)\bigg|_0^{\frac{\pi}{2}}=\frac{8a^2}{9}$$

Finally we use the formula (2.18) to compute the volume of a solid.

Example 3 Compute the volume of solid bounded by the cone $z = \sqrt{x^2+y^2}$ and paraboloid of revolution $z = 2 - x^2 - y^2$.

First we find the intersection of these two surfaces. The equation of the cone can be converted to

$$z^2 = x^2 + y^2$$

and substituting $x^2 + y^2$ into the equation of paraboloid we get $z = 2 - z^2$ or $z^2 + z - 2 = 0$.

This quadratic equation has two solutions $z_1 = 1$ and $z_2 = -2$. The second solution is impossible because of the equation of cone. Thus, these two surfaces intersect on the plane $z = 1$ and the intersection curve is the circle $x^2 + y^2 = 1$.

According to (2.18) the volume is

$$V = \iiint_V dxdydz$$

To evaluate this triple integral we use cylindrical coordinates. The projection of this solid onto xy-plane is the disk $x^2 + y^2 \le 1$. In cylindrical coordinates this disk is determined by inequalities $0 \le \varphi \le 2\pi$ and $0 \le \rho \le 1$. The surface on the top is paraboloid of revolution and the surface on the bottom is cone. In cylindrical coordinates $x^2 + y^2 = \rho^2$ thus, the equation of the cone is in cylindrical coordinates $z = \rho$ and the equation of the paraboloid is $z = 2-\rho^2$. Consequently, in this region $\rho \le z \le 2-\rho^2$ and our triple integral is in cylindrical coordinates

$$V = \iiint_V dxdydz = \int_0^{2\pi} d\varphi \int_0^1 d\rho \int_\rho^{2-\rho^2} \rho dz$$

Integration with respect to z gives

$$\rho \cdot z \Big|_\rho^{2-\rho^2} = \rho(2 - \rho^2 - \rho) = 2\rho - \rho^3 - \rho^2$$

Integration with respect to ρ gives

$$\int_0^1 (2\rho - \rho^3 - \rho^2) d\rho = \left(\rho^2 - \frac{\rho^4}{4} - \frac{\rho^3}{3}\right)\Big|_0^1 = \frac{5}{12}$$

and the volume of the solid is

$$V = \int_0^{2\pi} \frac{5}{12} d\varphi = \frac{5}{12}\varphi\Big|_0^{2\pi} = \frac{5\pi}{6}$$

3 Line and surface integrals

Line integral is an integral where the function to be integrated is evaluated along a curve. The terms path integral, curve integral, and curvilinear integral are also used.

3.1 Line integral with respect to arc length

Suppose that on the plane curve AB there is defined a function of two variables $f(x, y)$, i.e. to any point (x, y) of this curve there is related the value $f(x, y)$. Let

$$A = P_0,\ P_1,\ P_2,\ \ldots, P_{k-1},\ P_k,\ \ldots,\ P_n = B$$

the random partition of the curve AB into subarcs $\overset{\frown}{P_{k-1}P_k}$, $k = 1,\ 2,\ \ldots,\ n$. From every subarc we pick a random point $Q_k(\xi_k, \eta_k) \in \overset{\frown}{P_{k-1}P_k}$.

Denote by Δs_k the length of the subarc $\overset{\frown}{P_{k-1}P_k}$. Now we multiply the value at the point chosen by the length of subarc $f(Q_k)\Delta s_k$, where $k = 1,\ 2,\ \ldots,\ n$. Adding all those products, we get the sum

$$s_n = \sum_{k=1}^{n} f(Q_k)\Delta s_k \tag{3.26}$$

which is called the *integral sum* of the function $f(x, y)$ *over the curve* AB.

We have the random partition of the curve AB. Therefore, the lengths Δs_k of subarcs $\overset{\frown}{P_{k-1}P_k}$ are different. Denote by λ the greatest length of subarcs, i.e.

$$\lambda = \max_{1 \le k \le n} \Delta s_k$$

Definition. If there exists the limit

$$\lim_{\lambda \to 0} s_n$$

and this limit does not depend on the partition of AB and does not depend on the choice of the points Q_k on the subarcs, then this limit is called the *line integral with respect to arc length* and denoted by

$$\int_{AB} f(x, y)ds$$

Thus, by the definition

$$\int_{AB} f(x,y)ds = \lim_{\lambda\to 0}\sum_{k=1}^{n} f(Q_k)\Delta s_k$$

Line integral with respect to arc length is also referred as *line integral of a scalar field* because $f(x,y)$ defines a scalar field on the curve AB.

Suppose the curve AB is the piece of wire. If the function $\rho(x,y) \geq 0$ represents the density (mass per unit length) for wire AB, then the product $\rho(Q_k)\Delta s_k$ is the approximate mass of subarc Δs_k and the integral sum

$$\sum_{k=1}^{n} \rho(Q_k)\Delta s_k$$

is the approximate mass of the wire AB. For shorter subarc the value $\rho(Q_k)$ represents the variable density $\rho(x,y)$ of subarc with greater accuracy. Thus, in this case the limit of the integral sum, i.e. the line integral with respect to arc length gives the mass of the wire AB:

$$m = \int_{AB} \rho(x,y)ds \tag{3.27}$$

The properties on the line integral with respect to arc length can be proved directly, using the definition.

Property 1. The line integral with respect to arc length does not depend on the direction the curve AB has been traversed:

$$\int_{AB} f(x,y)ds = \int_{BA} f(x,y)ds$$

Property 2. (Additivity property) If C is some point on the curve AB, then

$$\int_{AB} f(x,y)ds = \int_{AC} f(x,y)ds + \int_{CB} f(x,y)ds$$

Property 3.

$$\int_{AB} [f(x,y) \pm g(x,y)]ds = \int_{AB} f(x,y)ds \pm \int_{AB} g(x,y)ds$$

Property 4. If c ic a constant, then

$$\int_{AB} cf(x,y)ds = c\int_{AB} f(x,y)ds$$

Property 5. Taking in the definition of the line integral with respect to arc length $f(x,y) \equiv 1$, we get the integral sum

$$s_n = \sum_{k=1}^{n} \Delta s_k$$

which is the sum of lengths of subarcs. This is the length of arc AB for any partition. Thus, for $f(x,y) \equiv 1$ the line integral gives us the length of arc AB:

$$s_{AB} = \int\limits_{AB} ds$$

Property 5 can be also obtained by taking in (3.27) the density $\rho(x,y) \equiv 1$ because then the mass and the length of the curve are numerically equal.

Any point of the curve AB in the space has three coordinates $Q_k(\xi_k, \eta_k, \zeta_k)$. So, the function defined on the space curve is in general a function of three variables $f(x,y,z)$. Defining the line integral with respect to arc length along the space curve we do everything like we did in the definition for the two-dimensional case:

$$\int\limits_{AB} f(x,y,z)ds = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(Q_k)\Delta s_k \tag{3.28}$$

Of course, five properties of the line integral for three-dimensional case are still valid.

3.2 Evaluation of line integral with respect to arc length

Suppose that the parametric equations of the curve AB in the plain are

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

and the parametric equations of the curve AB in the space are

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t), \end{cases}$$

where at the point A the value of the parameter $t = \alpha$ and at the point B the value of the parameter $t = \beta$.

Definition 1. The plain curve AB is called smooth, if $\dot{x} = \dfrac{dx}{dt}$ and $\dot{y} = \dfrac{dy}{dt}$ are continuous on $[\alpha; \beta]$ and

$$\dot{x}^2 + \dot{y}^2 \neq 0$$

Definition 2. The curve AB in the space is called smooth, if $\dot{x} = \dfrac{dx}{dt}$, $\dot{y} = \dfrac{dy}{dt}$ and $\dot{z} = \dfrac{dz}{dt}$ are continuous on $[\alpha; \beta]$ and

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \neq 0$$

Intuitively, a smooth curve is one that does not have sharp corners.

Theorem 1. If the function $f(x, y)$ is continuous on the smooth curve AB, then

$$\int\limits_{AB} f(x, y)ds = \int\limits_{\alpha}^{\beta} f[x(t), y(t)]\sqrt{\dot{x}^2 + \dot{y}^2}dt \tag{3.29}$$

Theorem 2. If the function $f(x, y, z)$ is continuous on the smooth curve AB, then

$$\int\limits_{AB} f(x, y, z)ds = \int\limits_{\alpha}^{\beta} f[x(t), y(t), z(t)]\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}dt \tag{3.30}$$

If $\mathbf{r}(t) = (x(t), y(t), z(t))$ is the position vector of a point on the curve, then the square root in the formula (3.30) is the length of $\dot{\mathbf{r}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$ i.e $|\dot{\mathbf{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. The formula (3.30) can be re-written as

$$\int\limits_{AB} f(x, y, z)ds = \int\limits_{\alpha}^{\beta} f[x(t), y(t), z(t)]|\dot{\mathbf{r}}(t)|dt$$

Suppose the curve AB is a graph of the function $y = \varphi(x)$ given explicitly, at the point A $x = a$ and at B $x = b$. The curve is smooth, if there exists $\varphi'(x)$ on the interval $[a; b]$.

Theorem 3. If the function $f(x, y)$ is continuous on the smooth curve AB, then

$$\int\limits_{AB} f(x, y)ds = \int\limits_{a}^{b} f[x, \varphi(x)]\sqrt{1 + y'^2}dx \tag{3.31}$$

This theorem is the direct conclusion of Theorem 1 because treating the variable x as the parameter, we have $\dot{x} = 1$ and $\dot{y} = \dfrac{dy}{dx} = y'$.

Example 1. Compute the line integral $\displaystyle\int\limits_{AB} \frac{ds}{x-y}$, where AB is the segment of the line $y = 2x - 3$ between coordinate axes.

The line is the graph of the function given explicitly. Therefore, we use for the computation the formula (3.31).

At the intersection point by y axis $x = 0$ and at the intersection point by x axis $y = 0$, i.e. $x = \dfrac{3}{2}$. To apply the formula, we find $y = 2$ and $1 + y'^2 = 5$. Thus,

$$\begin{aligned}\int\limits_{AB} \frac{ds}{x-y} &= \int\limits_0^{\frac{3}{2}} \frac{\sqrt{5}dx}{x-(2x-3)} = \sqrt{5}\int\limits_0^{\frac{3}{2}} \frac{dx}{3-x} = -\sqrt{5}\int\limits_0^{\frac{3}{2}} \frac{d(3-x)}{3-x} \\ &= -\sqrt{5}\ln|3-x|\Big|_0^{\frac{3}{2}} = -\sqrt{5}\left(\ln\frac{3}{2} - \ln 3\right) = -\sqrt{5}\ln\frac{1}{2} = \sqrt{5}\ln 2\end{aligned}$$

Example 2. Compute the line integral $\displaystyle\int\limits_{AB} \sqrt{y}ds$, where AB is the first arc of cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

For the first arc of cycloid $0 \le t \le 2\pi$. To apply the formula (3.29), we find $\dot{x} = a(1 - \cos t)$, $\dot{y} = a\sin t$ and

$$\dot{x}^2+\dot{y}^2 = a^2(1-\cos t)^2+a^2\sin^2 t = a^2(1-2\cos t+\cos^2 t+\sin^2 t) = 2a^2(1-\cos t)$$

By the formula (3.29)

$$\int\limits_{AB} \sqrt{y}ds = \int\limits_0^{2\pi} \sqrt{a(1-\cos t)}\sqrt{2a^2(1-\cos t)}dt =$$

$$a\sqrt{2a}\int\limits_0^{2\pi}(1-\cos t)dt = a\sqrt{2a}(t-\sin t)\Big|_0^{2\pi} = 2\pi a\sqrt{2a}$$

Example 3. Compute the line integral $\displaystyle\int\limits_{AB} (2z - \sqrt{x^2+y^2})ds$, where AB is the first turn of conical helix $x = t\cos t$, $y = t\sin t$, $z = t$.

For the first turn of conical helix $0 \le t \le 2\pi$. Find $\dot{x} = \cos t - t\sin t$, $\dot{y} = \sin t + t\cos t$, $\dot{z} = 1$ and

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1 =$$
$$\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 1 = 2 + t^2$$

By the formula (3.30) we obtain

$$\int\limits_{AB} (2z - \sqrt{x^2+y^2})ds = \int\limits_0^{2\pi} (2t - \sqrt{t^2\cos^2 t + t^2\sin^2 t})\sqrt{2+t^2}dt =$$
$$\int\limits_0^{2\pi} (2t - t)\sqrt{2+t^2}dt = \int\limits_0^{2\pi} t\sqrt{2+t^2}dt = \frac{1}{2}\int\limits_0^{2\pi} \sqrt{2+t^2}d(2+t^2) =$$
$$\frac{1}{2}\frac{(2+t^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{2\pi} = \frac{(2+t^2)^{\frac{3}{2}}}{3}\Bigg|_0^{2\pi} = \frac{(2+4\pi^2)\sqrt{2+4\pi^2} - 2\sqrt{2}}{3}$$

3.3 Line integral with respect to coordinates

In the first subsection we defined the line integral for the scalar field. Now we are going to define the line integral for the vector field. First we consider the two-dimensional case. Let AB be the curve in the plain and $\overrightarrow{F} = (X(x,y); Y(x,y))$ a force vector. Suppose that the force is applied to an object to move it along the curve AB. The goal in to find the work done by this force. To do it, we first divide the curve AB with the points

$$A = P_0,\ P_1,\ \ldots,\ P_{k-1},\ P_k,\ \ldots,\ P_n = B$$

into subarcs $\overset{\frown}{P_{k-1}P_k}$, where $k = 1,\ 2,\ \ldots,\ n$ and approximate any subarc $\overset{\frown}{P_{k-1}P_k}$ to the vector $\overrightarrow{P_{k-1}P_k}$.

Denote the coordinates of the kth partition point P_k by x_k and y_k, i.e. $P_k(x_k; y_k)$ and the coordinates of the vector $\overrightarrow{P_{k-1}P_k}$ by

$$\Delta x_k = x_k - x_{k-1}$$

and

$$\Delta y_x = y_k - y_{k-1}$$

that is

$$\overrightarrow{P_{k-1}P_k} = (\Delta x_k; \Delta y_k)$$

Let Δs_k be the magnitude of the vector $\overrightarrow{P_{k-1}P_k}$:

$$\Delta s_k = \sqrt{\Delta x_k^2 + \Delta y_k^2}$$

and λ the greatest of all those magnitudes

$$\lambda = \max_{1 \le k \le n} \Delta s_k$$

Next we choose a random point $Q_k(\xi_k; \eta_k)$ on any subarc $\overset{\frown}{P_{k-1}P_k}$ and substitute on this subarc the variable force vector by the constant force vector

$$\overrightarrow{F_k} = (X(\xi_k, \eta_k); Y(\xi_k, \eta_k))$$

Recall that if a constant force $\overrightarrow{F_k}$ is applied to an object to move it along a straight line from the point P_{k-1} to the point P_k, then the amount of work done A_k is the scalar product of the force vector and the vector $\overrightarrow{P_{k-1}P_k}$:

$$A_k = \overrightarrow{F_k} \cdot \overrightarrow{P_{k-1}P_k} = X(\xi_k, \eta_k)\Delta x_k + Y(\xi_k, \eta_k)\Delta y_k$$

The total work done by the force vector $\overrightarrow{F}$, moving an object from the point A to the point B along the curve is approximately

$$\sum_{k=1}^{n} [X(\xi_k, \eta_k)\Delta x_k + Y(\xi_k, \eta_k)\Delta y_k]. \tag{3.32}$$

Approximately because we have approximated the subarc $\overset{\frown}{P_{k-1}P_k}$ to the vector $\overrightarrow{P_{k-1}P_k}$ and the variable force vector $\overrightarrow{F} = (X(x, y); Y(x, y))$ to the constant vector $\overrightarrow{F_k} = (X(\xi_k, \eta_k); Y(\xi_k, \eta_k))$.

Obviously, taking more partition points, the subarcs get shorter and the vectors $\overrightarrow{P_{k-1}P_k}$ represent the subarcs $\overset{\frown}{P_{k-1}P_k}$ with greater accuracy. As well, the constant vector $\overrightarrow{F_k} = (X(\xi_k, \eta_k); Y(\xi_k, \eta_k))$ represents the variable vector $\overrightarrow{F} = (X(x, y); Y(x, y))$ on $\overset{\frown}{P_{k-1}P_k}$ with greater accuracy.

Definition. If the sum (3.32) has the limit as $\max \Delta s_k \to 0$ and this limit does not depend on the partition of the curve AB and does not depend on the choice of points Q_k on subarcs, then this limit is called the *line integral with respect to coordinates* and denoted

$$\int_{AB} X(x, y)dx + Y(x, y)dy$$

Thus, by the definition

$$\int_{AB} X(x, y)dx + Y(x, y)dy = \lim_{\lambda \to 0} \sum_{k=1}^{n} [X(\xi_k, \eta_k)\Delta x_k + Y(\xi_k, \eta_k)\Delta y_k] \tag{3.33}$$

If AB is a curve in the space, then

$$\overrightarrow{P_{k-1}P_k} = (\Delta x_k; \Delta y_k; \Delta z_k)$$

and the magnitude of this vector

$$\Delta s_k = \sqrt{\Delta x_k^2 + \Delta y_k^2 + \Delta z_k^2}$$

Also the force vector has three coordinates

$$\overrightarrow{F} = (X(x,y,z); Y(x,y,z)); Z(x,y,z))$$

The line integral with respect to coordinates is defined as the limit

$$\int\limits_{AB} X(x,y,z)dx + Y(x,y,z)dy + Z(x,y,z)dz$$

$$= \lim_{\lambda \to 0} \sum_{k=1}^{n} [X(\xi_k, \eta_k, \zeta_k)\Delta x_k + Y(\xi_k, \eta_k, \zeta_k)\Delta y_k + Z(\xi_k, \eta_k, \zeta_k)\Delta z_k]$$

We consider the properties of the line integral with respect to coordinates for the curve in the plane. All of this discussion generalizes to space curves in a straightforward manner.

Property 1. If C is a random point on the curve AB, then

$$\int\limits_{AB} X(x,y)dx+Y(x,y)dy = \int\limits_{AC} X(x,y)dx+Y(x,y)dy+\int\limits_{CB} X(x,y)dx+Y(x,y)dy \tag{3.34}$$

Property 2. If the curve is traced in reverse (that is, from the terminal point to the initial point), then the sign of the line integral is reversed as well:

$$\int\limits_{BA} X(x,y)dx + Y(x,y)dy = -\int\limits_{AB} X(x,y)dx + Y(x,y)dy \tag{3.35}$$

3.4 Evaluation of line integral with respect to coordinates

Suppose that AB is a smooth curve in the plain

$$x = x(t), \;\; y = y(t)$$

and the functions $X(x, y)$ and $Y(x, y)$ are continuous on AB. Let at the point A the parameter $t = \alpha$ and at the point B $t = \beta$.

Theorem 1. If the functions $X(x, y)$ and $Y(x, y)$ are continuous on the smooth curve AB, then

$$\int\limits_{AB} X(x, y)dx + Y(x, y)dy = \int\limits_{\alpha}^{\beta} [X(x(t), y(t))\dot{x} + Y(x(t), y(t))\dot{y}]dt \tag{3.36}$$

In three dimensional case there holds the similar theorem. Suppose that on the line AB

$$x = x(t), \;\; y = y(t), \;\; z = z(t)$$

there is defined a vector function $\overrightarrow{F}(x, y, z) = X(x, y, z), Y(x, y, z), Z(x, y, z)$. Suppose again that at the point A the parameter $t = \alpha$ and at the point B $t = \beta$.

Theorem 2. If the functions $X(x, y, z)$, $Y(x, y, z)$ and $Z(x, y, z)$ are continuous on the smooth curve AB, then

$$\begin{aligned}&\int\limits_{AB} X(x, y, z)dx + Y(x, y, z)dy + Z(x, y, z)dz \\ &= \int\limits_{\alpha}^{\beta} [X(x(t), y(t), z(t))\dot{x} + Y(x(t), y(t), z(t))\dot{y} + Z(x(t), y(t), z(t))\dot{z}]dt\end{aligned} \tag{3.37}$$

Conclusion. Suppose the plain curve AB is the graph of the function $y = y(x)$ given explicitly and at the point A $x = a$ and at B $x = b$. Treating the variable x as a parameter, we obtain $\dot{x} = 1$, $\dot{y} = y'$ and by the formula (3.36)

$$\int\limits_{AB} X(x, y)dx + Y(x, y)dy = \int\limits_{a}^{b} [X(x, y(x)) + Y(x, y(x))y']dx \tag{3.38}$$

Remark. Sometimes (especially for vertical lines) it is necessary to consider y as the independent variable and x as the function $x = x(y)$. Changing the roles of the variables x and y, we get

$$\int\limits_{AB} X(x, y)dx + Y(x, y)dy = \int\limits_{a}^{b} [X(x(y), y)x' + Y(x(y), y)]dy \tag{3.39}$$

A curve L is called *closed* if its initial and final points are the same point. For example a circle is a closed curve. A curve L is called *simple* if it doesn't cross itself. A circle is a simple curve while a figure ∞ type curve is not simple. If L is not a smooth curve, but can be broken into a finite number of smooth curves, then we say that L is *piecewise smooth*. The line integral over the piecewise smooth closed simple curve L is often denoted

$$\oint\limits_L X(x,y)dx + Y(x,y)dy$$

The *positive orientation* of the closed curve L is that as we traverse the curve following the positive orientation the region D bounded by L must always be on the left.

Example 1. Compute $\int\limits_{AB} x\cos y dx - y\sin x dy$ over the straight line from $A(0;0)$ to $B(\pi;2\pi)$.

The direction vector of the line is $\overrightarrow{AB} = (\pi;2\pi)$ and the parametric equations

$$x = \pi t$$
$$y = 2\pi t,$$

At the point A the parameter $t = 0$ and at the point B $t = 1$. To apply the formula (3.36) we find $\dot{x} = \pi$ and $\dot{y} = 2\pi$. By the formula

$$\int\limits_{AB} x\cos y dx - y\sin x dy = \int\limits_0^1 (\pi t\cos 2\pi t\cdot\pi - 2\pi t\sin\pi t\cdot 2\pi)dt$$

$$= \pi^2\int\limits_0^1 [t(\cos 2\pi t - 4\sin\pi t)]dt = \dots$$

The integral obtained we integrate by parts, taking

$$u = t, \qquad dv = \cos 2\pi t - 4\sin\pi t$$

Then

$$du = dt, \qquad v = \frac{1}{2\pi}\sin 2\pi t + \frac{4}{\pi}\cos\pi t$$

and

$$\ldots = \pi^2 \left[t \left(\frac{1}{2\pi} \sin 2\pi t + \frac{4}{\pi} \cos \pi t \right) \Big|_0^1 - \int\limits_0^1 \left(\frac{1}{2\pi} \sin 2\pi t + \frac{4}{\pi} \cos \pi t \right) dt \right]$$

$$= \pi^2 \left[-\frac{4}{\pi} + \left(\frac{1}{4\pi^2} \cos 2\pi t - \frac{4}{\pi^2} \sin \pi t \right) \Big|_0^1 \right] = -4\pi$$

Example 2. Compute $\oint\limits_L (x^2 + y)dx + xydy$, where L is the positively oriented triangle OAB with vertices $O(0;0)$, $A(2;1)$ and $B(0;1)$.

The triangle is sketched in Figure 7.3. Notice that the triangle is a simple closed piecewise smooth curve, because it consists of three smooth lines.

By Property 1

$$\oint\limits_L (x^2+y)dx+xydy = \int\limits_{OA} (x^2+y)dx+xydy+\int\limits_{AB} (x^2+y)dx+xydy+\int\limits_{BO} (x^2+y)dx+xydy$$

By Property 2 the direction is important. Compute all three line integrals. The side OA has the equation $y = \frac{x}{2}$, $0 \leq x \leq 2$ and $y' = \frac{1}{2}$. By the formula (3.38)

$$\int\limits_{OA} (x^2 + y)dx + xydy = \int\limits_0^2 \left(x^2 + \frac{x}{2} + x \cdot \frac{x}{2} \cdot \frac{1}{2} \right) dx = \int\limits_0^2 \left(\frac{5x^2}{4} + \frac{x}{2} \right) dx$$

The side AB has the equation $y = 1$, hence, $y' = 0$. At the initial point A $x = 2$ and at the end point B $x = 0$. Thus, by (3.38)

$$\int\limits_{AB} (x^2 + y)dx + xydy = \int\limits_2^0 (x^2 + 1 + x \cdot 1 \cdot 0)dx = \int\limits_2^0 (x^2 + 1)dx$$

The third side BO of the triangle is the vertical line $x = 0$, hence, $x' = 0$. At the point B $y = 1$ and at the point O $y = 0$. To compute the third line integral we use the formula (3.39)

$$\int\limits_{BO} (x^2+y)dx + xydy = \int\limits_{1}^{0} [(0+y)\cdot 0 + 0\cdot y]dy = 0$$

Therefore,

$$\oint\limits_{L} (x^2+y)dx + xydy = \int\limits_{0}^{2} \left(\frac{5x^2}{4} + \frac{x}{2}\right) dx + \int\limits_{2}^{0} (x^2+1)dx$$

Changing the limits in the last integral gives

$$\oint\limits_{L} (x^2+y)dx + xydy = \int\limits_{0}^{2} \left(\frac{5x^2}{4} + \frac{x}{2} - x^2 - 1\right) dx$$

$$= \int\limits_{0}^{2} \left(\frac{x^2}{4} + \frac{x}{2} - 1\right) dx = \left(\frac{x^3}{12} + \frac{x^2}{4} - x\right)\Bigg|_0^2 = \frac{2}{3} + 1 - 2 = -\frac{1}{3}$$

We shall return to the last example once more.

3.5 Green's formula

In this subsection we are going to investigate the relationship between certain kinds of line integrals (on closed curves) and double integrals. Suppose the functions $X(x,y)$ and $Y(x,y)$ are defined on the simple closed curve L and in the region D enclosed by this curve.

Theorem (Green's formula). If the functions $X(x,y)$ and $Y(x,y)$ are continuous on the closed simple piecewise smooth curve L, the partial derivatives $\dfrac{\partial Y}{\partial x}$ and $\dfrac{\partial X}{\partial y}$ are continuous in the regular region D and L is positively oriented, then

$$\oint\limits_{L} X(x,y)dx + Y(x,y)dy = \iint\limits_{D} \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dxdy \tag{3.40}$$

Example. Let us compute the line integral

$$\oint_L (x^2+y)dx + xydy$$

given in Example 2 of the previous subsection once more, using the Green's formula.

Here $X(x,y) = x^2+y$ and $Y(x,y) = xy$. To apply the Green's formula (3.40) we find $\dfrac{\partial Y}{\partial x} = y$ and $\dfrac{\partial X}{\partial y} = 1$. Let D be the region bounded by L. By the formula (3.40)

$$\oint_L (x^2+y)dx + xydy = \iint_D (y-1)dxdy$$

Using Figure 7.3, we determine the limits of integration $0 \le x \le 2$ and $\dfrac{x}{2} \le y \le 1$. Hence,

$$\oint_L (x^2+y)dx + xydy = \int_0^2 dx \int_{\frac{x}{2}}^1 (y-1)dy$$

Find the inside integral

$$\int_{\frac{x}{2}}^1 (y-1)dy = \int_{\frac{x}{2}}^1 (y-1)d(y-1) = \left.\frac{(y-1)^2}{2}\right|_{\frac{x}{2}}^1 = -\frac{\left(\frac{x}{2}-1\right)^2}{2} = -\frac{(x-2)^2}{8}$$

and the outside integral

$$\int_0^2 \left[-\frac{(x-2)^2}{8}\right] dx = -\frac{1}{8}\int_0^2 (x-2)^2 d(x-2) = -\frac{1}{8}\left.\frac{(x-2)^3}{3}\right|_0^2 = \frac{1}{8}\frac{(-2)^3}{3} = -\frac{1}{3}$$

3.6 Path independent line integral

In this subsection we find out in what conditions the line integral

$$\int_{AB} X(x,y)dx + Y(x,y)dy \tag{3.41}$$

depends only on the endpoints A and B of the line but not on the path of integration.

Assume that in the region D containing the points A and B the functions $X(x, y)$ and $Y(x, y)$ and the partial derivatives $\dfrac{\partial X}{\partial y}$ and $\dfrac{\partial Y}{\partial x}$ are continuous. Let's choose two whatever curves AEB and AFB in the region D joining the points A and B.

So, we want to know in which conditions for any curves AEB and AFB

$$\int\limits_{AEB} Xdx + Ydy = \int\limits_{AFB} Xdx + Ydy$$

i.e.

$$\int\limits_{AEB} Xdx + Ydy - \int\limits_{AFB} Xdx + Ydy = 0$$

By Property 2 of the line integral with respect to coordinates

$$\int\limits_{AEB} Xdx + Ydy + \int\limits_{BFA} Xdx + Ydy = 0$$

and by Property 1

$$\int\limits_{AEBFA} Xdx + Ydy = 0$$

Denoting the closed curve $AEBFA = L$, we obtain the condition

$$\oint\limits_{L} Xdx + Ydy = 0 \tag{3.42}$$

This condition we obtain for any curves between any two points A and B in the region D. We shall call the curve joining the points A and B the *path of integration.*

Consequently, if the line integral (3.41) is path independent, then for each closed curve L in the region D there holds (3.42).

Theorem 1. The line integral (3.41) is path independent in the region D if and only if for any closed curve L in the region D there holds (3.42).

Next, suppose that for every closed curve L in the region D there holds (3.42). By the assumptions made in the beginning of this subsection there holds Green's formula. Denote by Δ the region enclosed by the closed curve L. According to Green's formula (3.40)

$$\iint\limits_{\Delta} \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dxdy = 0$$

Then also

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = 0$$

or

$$\frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y} \tag{3.43}$$

Now Theorem 1 gives us the following theorem.

Theorem 2. The line integral (3.41) is path independent in the region D if and only if in the region D there holds the condition (3.43).

The path independent line integral (3.41) is also denoted by

$$\int\limits_{A}^{B} X dx + Y dy$$

Example 1. The line integral

$$\int\limits_{A}^{B} (2x \cos y - y^2 \sin x) dx + (2y \cos x - x^2 \sin y) dy$$

is path independent because

$$\frac{\partial}{\partial x}(2y \cos x - x^2 \sin y) = -2y \sin x - 2x \sin y$$

and

$$\frac{\partial}{\partial y}(2x \cos y - y^2 \sin x) = -2x \sin y - 2y \sin x$$

Example 2. Compute

$$\int\limits_{(0,0)}^{(2,1)} 2xy dx + x^2 dy$$

This line integral is path independent because

$$\frac{\partial (x^2)}{\partial x} = 2x$$

and

$$\frac{\partial(2xy)}{\partial y} = 2x$$

Thus, we can choose whatever path of integration joining the points $(0;0)$ and $(2;1)$. Let's choose the broken line OBA, where $O(0,0)$, $B(2;0)$ and $A(2;1)$. Usually, choosing the kind of broken line, whose segments are parallel to coordinate axes, gives us the most simple computation.

By Property 1 of the line integral with respect to coordinates

$$\int\limits_{(0,0)}^{(2,1)} 2xydx + x^2dy = \int\limits_{(0,0)}^{(2,0)} 2xydx + x^2dy + \int\limits_{(2,0)}^{(2,1)} 2xydx + x^2dy$$

The equation of the line OB is $y = 0$, which gives $y' = 0$. On the segment OB $0 \leq x \leq 2$ and by the formula (3.38)

$$\int\limits_{(0,0)}^{(2,0)} 2xydx + x^2dy = \int\limits_{0}^{2} (2x \cdot 0 + x^2 \cdot 0)dx = 0$$

The equation of the line BA is $x = 2$, i.e. $x' = 0$. On the segment BA the variable $0 \leq y \leq 1$ and by the formula (3.39)

$$\int\limits_{(2,0)}^{(2,1)} 2xydx + x^2dy = \int\limits_{0}^{1} (4y \cdot 0 + 4)dy = 4$$

Hence,

$$\int\limits_{(0,0)}^{(2,1)} 2xydx + x^2dy = 4$$

If there exists a function of two variables $u(x,y)$ such that the total differential of this function is

$$du = X(x,y)dx + Y(x,y)dy$$

i.e. $X = \dfrac{\partial u}{\partial x}$ and $Y = \dfrac{\partial u}{\partial y}$, then

$$\frac{\partial X}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}$$

and

$$\frac{\partial Y}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}$$

Because of continuity the condition (3.43) holds.

Recall that the vector field $\overrightarrow{F} = (X(x,y), Y(x,y))$ is conservative, if $\overrightarrow{F}$ is the gradient of a scalar field $u(x,y)$ and the function $u(x,y)$ is the potential function of $\overrightarrow{F}$. Then $du = X(x,y)dx + Y(x,y)dy$ is the total differential of $u(x,y)$ and the condition (3.43) holds.

Conclusion 1. For the conservative vector field $\overrightarrow{F} = (X(x,y), Y(x,y))$ the line integral (3.41) is path independent.

Conclusion 2. For the conservative vector field $\overrightarrow{F} = (X(x,y), Y(x,y))$ the line integral over any closed curve L

$$\oint\limits_L X(x,y)dx + Y(x,y)dy = 0$$

Conclusion 3. If $u(x,y)$ is the potential function of the conservative vector field $\overrightarrow{F} = (X(x,y), Y(x,y))$, then

$$\int\limits_A^B X(x,y)dx + Y(x,y)dy = \int\limits_A^B du(x,y) = u(x,y)\Big|_A^B$$

3.7 Surface integral of scalar fields

In mathematical analysis, a surface integral is a generalization of multiple integrals to integration over surfaces. It is like the double integral analog of the line integral. One may integrate over given surface scalar fields and vector fields. Let's start from the integration scalar fields over surface.

Suppose that the function of three variables $f(x,y,z)$ is defined on the surface S in the xyz axes.

Choose whatever partition of the surface S into n subsurfaces $\Delta\sigma_k$ $(1 \leq$

$k \leq n$), where $\Delta\sigma_k$ denotes the kth subsurface as well as its area.

On any of these subsurfaces we pick a random point $P_k(\xi_k; \eta_k; \zeta_k) \in \Delta\sigma_k$ and find the products

$$f(P_k)\Delta\sigma_k$$

Adding those products, we get the *integral sum* of the function $f(x, y, z)$ *over the surface* S

$$\sum_{k=1}^{n} f(P_k)\Delta\sigma_k$$

The greatest distance between the points on the subsurface is called the diameter of the subsurface diam $\Delta\sigma_k$. Every subsurface has its own diameter. In general those diameters are different because we have the random partition of the surface S. Denote the greatest diameter by λ, i.e.

$$\lambda = \max_{1\leq k\leq n} \operatorname{diam} \Delta\sigma_k$$

Definition 1. If there exists the limit

$$\lim_{\lambda\to 0} \sum_{k=1}^{n} f(P_k)\Delta\sigma_k$$

and this limit does not depend on the partition of the surface S and does not depend on the choice of points P_k on the subsurfaces, then this limit is called the *surface integral with respect to area of surface* and denoted

$$\iint\limits_S f(x, y, z)d\sigma$$

By Definition 1

$$\iint\limits_S f(x, y, z)d\sigma = \lim_{\lambda\to 0} \sum_{k=1}^{n} f(P_k)\Delta\sigma_k$$

Sometimes the surface integral with respect to area of surface is referred as the *surface integral of the scalar field.* The properties of the surface integral with respect to area of surface are familiar already. While formulating the properties, we use the term "surface integral"and "with respect to area of surface"will be omitted.

Property 1. The surface integral of the sum (difference) of two functions equals to the sum (difference) of surface integrals of these functions:

$$\iint\limits_S [f(x, y, z) \pm g(x, y, z)]d\sigma = \iint\limits_S f(x, y, z)d\sigma \pm \iint\limits_S g(x, y, z)d\sigma$$

Property 2. The constant factor can be taken outside the surface integral, i.e. if c is a constant then

$$\iint\limits_S cf(x,y,z)d\sigma = c\iint\limits_S f(x,y,z)d\sigma$$

Property 3. If the surface is the unit of two surfaces, $S = S_1 \cup S_2$ and S_1 and S_2 have no common interior point, then

$$\iint\limits_S f(x,y,z)d\sigma = \iint\limits_{S_1} f(x,y,z)d\sigma + \iint\limits_{S_2} f(x,y,z)d\sigma$$

Suppose the surface S is the graph of the function of two variables $z = z(x,y)$. Denote by D the projection of the surface S onto xy plane. The surface S is called *smooth* if the function $z(x,y)$ has continuous partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in D.

The following theorem gives the formula to evaluate the surface integral with respect to area of surface.

Theorem. If the function $f(x,y,z)$ is continuous on the smooth surface S and D is the projection of S onto xy plane, then

$$\iint\limits_S f(x,y,z)d\sigma = \iint\limits_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy \quad (3.44)$$

Thus, in order to evaluate a surface integral we will substitute the equation of the surface in for z in the integrand and then add on the factor square root. After that the integral is a standard double integral and by this point we should be able to deal with that.

If the function $f(x,y,z) \equiv 1$ on the surface S, then the formula

$$\iint\limits_S d\sigma = \iint\limits_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy \quad (3.45)$$

gives us the area of the surface S.

Example 1. Evaluate $\iint\limits_S (x^2+y^2+z^2)d\sigma$, if S is the portion of the cone $z = \sqrt{x^2+y^2}$, where $0 \le z \le 1$.

The plane $z = 1$ and the cone $z = \sqrt{x^2+y^2}$ intersect along the circle

$$x^2+y^2 = 1$$

The projection of the portion of the cone onto xy plane is the disk $x^2+y^2 \leq 1$.

To apply the formula (3.44) we find

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

and

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}} = \sqrt{2}$$

By the formula (3.44)

$$\iint\limits_S (x^2+y^2+z^2)d\sigma = \iint\limits_D (x^2+y^2+x^2+y^2)\sqrt{2}dxdy = 2\sqrt{2}\iint\limits_D (x^2+y^2)dxdy$$

The region of integration D in the double integral obtained is the disk of radius 1 centered at the origin. To compute this double integral we convert it into polar coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. Then $x^2+y^2 = \rho^2$ and $|J| = \rho$.

The region of integration in polar coordinates is determined by inequalities $0 \leq \varphi \leq 2\pi$ and $0 \leq \rho \leq 1$. Hence,

$$2\sqrt{2}\iint\limits_D (x^2+y^2)dxdy = 2\sqrt{2}\int_0^{2\pi} d\varphi \int_0^1 \rho^2\rho d\rho$$

First we compute the inside integral

$$\int_0^1 \rho^3 d\rho = \frac{1}{4}$$

and finally the outside integral

$$2\sqrt{2}\int_0^{2\pi}\frac{1}{4}d\varphi = \frac{\sqrt{2}}{2}\int_0^{2\pi} d\varphi = \pi\sqrt{2}$$

Example 2. Compute the area of the portion of paraboloid of rotation $z = x^2 + y^2$ under the plane $z = 4$.

The projection D of the portion of paraboloid of rotation onto xy plane is the disk $x^2 + y^2 \leq 4$ of radius 2 centered at the origin.we find

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y} = 2y$$

and

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

Thus, by the formula (3.45) the area of the portion of paraboloid of rotation is

$$\iint\limits_S d\sigma = \iint\limits_D \sqrt{1 + 4x^2 + 4y^2}dxdy$$

The double integral obtained we convert to polar coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. Then $1 + 4x^2 + 4y^2 = 1 + 4\rho^2$ and $|J| = \rho$ and the region D is determined by $0 \leq \varphi \leq 2\pi$ and $0 \leq \rho \leq 2$. Therefore,

$$\iint\limits_D \sqrt{1 + 4x^2 + 4y^2}dxdy = \int\limits_0^{2\pi} d\varphi \int\limits_0^2 \sqrt{1 + 4\rho^2}\rho d\rho$$

To find the inside integral we use the equality of differentials $d(1 + 4\rho^2) = 8\rho d\rho$, which gives

$$\begin{aligned}\int\limits_0^2 \sqrt{1 + 4\rho^2}\rho d\rho &= \frac{1}{8}\int\limits_0^2 \sqrt{1 + 4\rho^2}8\rho d\rho \\ &= \frac{1}{8}\int\limits_0^2 (1 + 4\rho^2)^{\frac{1}{2}} d(1 + 4\rho^2) = \frac{1}{8}\left.\frac{(1 + 4\rho^2)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^2 \\ &= \left.\frac{1}{12}(1 + 4\rho^2)\sqrt{1 + 4\rho^2}\right|_0^2 = \frac{17\sqrt{17} - 1}{12}\end{aligned}$$

The outside integral, i.e. the area to be computed is

$$\frac{17\sqrt{17} - 1}{12}\int\limits_0^{2\pi} d\varphi = \frac{17\sqrt{17} - 1}{12} \cdot 2\pi = \frac{\pi(17\sqrt{17} - 1)}{6}$$

3.8 Surface integral with respect to coordinates

Suppose that S is a surface in the space and let $Z(x,y,z)$ be a function defined at all points of S. Choose a whatever partition of the surface S into n nonoverlapping subsurfaces $\Delta\sigma_k$ $(1 \leq k \leq n)$. In any of these subsurfaces we pick a random point $P_k(\xi_k;\eta_k;\zeta_k)$ and compute the value of function $Z(P_k)$. Let us denote by Δs_k the projection of $\Delta\sigma_k$ onto xy plane, where Δs_k denotes also the area of this projection. Next we find the products $Z(P_k)\Delta s_k$ and adding these products, we get the sum

$$\sum_{k=1}^{n} Z(P_k)\Delta s_k$$

which is called the integral sum of the function $Z(x,y,z)$ over the projection of surface S onto xy plane. Let $\operatorname{diam}\Delta s_k$ be the diameter of Δs_k. We have a random partition of the surface S, hence the diameters of these projections are different. Denote by λ the greatest diameter of the projections of subsurfaces $\Delta\sigma_k$, i.e.

$$\lambda = \max_{1\leq k\leq n} \operatorname{diam}\Delta s_k$$

Definition 1. If there exists the limit

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} Z(P_k)\Delta s_k$$

and this limit does not depend on the partition of the surface S and it is independent on the choice of points P_k in the subsurfaces, then this limit is called the *surface integral* of the function $Z(x,y,z)$ *over the projection of the surface* onto xy plane and denoted

$$\iint_S Z(x,y,z)dxdy$$

Thus, by the definition

$$\iint_S Z(x,y,z)dxdy = \lim_{\lambda\to 0}\sum_{k=1}^{n} Z(P_k)\Delta s_k \tag{3.46}$$

Second, suppose that the function of three variables $Y(x,y,z)$ is defined at all points of the surface S and that $\Delta s'_k$ is the projection of $\Delta\sigma_k$ onto xz plane. Choosing again a random point $P_k \in \Delta\sigma_k$, we find the products $Y(P_k)\Delta s'_k$. The sum of these products

$$\sum_{k=1}^{n} Y(P_k)\Delta s'_k$$

is called the integral sum of the function $Y(x, y, z)$ over the projection of S onto xz plane.

Definition 2. If there exists the limit

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} Y(P_k)\Delta s'_k$$

and this limit does not depend on the partition of the surface S and it is independent on the choice of points P_k in the subsurfaces, then this limit is called the *surface integral* of the function $Y(x, y, z)$ *over the projection of the surface* onto xz plane and denoted

$$\iint_S Y(x, y, z)dxdz$$

By Definition 2

$$\iint_S Y(x, y, z)dxdz = \lim_{\lambda \to 0} \sum_{k=1}^{n} Y(P_k)\Delta s'_k \tag{3.47}$$

Third, suppose that the function of three variables $X(x, y, z)$ is defined at all points of the surface S and $\Delta s''_k$ is the projection of $\Delta\sigma_k$ onto yz plane. We choose again random points $P_k \in \Delta\sigma_k$ and find the products $X(P_k)\Delta s''_k$. The sum

$$\sum_{k=1}^{n} X(P_k)\Delta s''_k$$

is called the integral sum of function $X(x, y, z)$ over the projection of S onto yz plane.

Definition 3. If there exists the limit

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} X(P_k)\Delta s''_k$$

and this limit does not depend on the partition of the surface S and does not depend on the choice of points P_k in the subsurfaces, then this limit is called the *surface integral* of the function $X(x, y, z)$ *over the projection of the surface* onto yz plane and denoted

$$\iint_S X(x, y, z)dydz$$

By Definition 3

$$\iint_S X(x,y,z)dydz = \lim_{\lambda\to 0}\sum_{k=1}^{n} X(P_k)\Delta s_k'' \tag{3.48}$$

In general we define the surface integral over the projection of the vector function

$$\overrightarrow{F}(x,y,z) = (X(x,y,z); Y(x,y,z); Z(x,y,z))$$

as

$$\iint_S X(x,y,z)dydz + Y(x,y,z)dxdz + Z(x,y,z)dxdy \tag{3.49}$$

Remark. Sometimes the surface integral over the projection is also referred as the *surface integral of the vector field.*

3.9 Evaluation of surface integral over the projection

Consider the evaluation of the surface integral over the projection onto xy plane

$$\iint_S Z(x,y,z)dxdy$$

Suppose that the smooth surface S is a graph of the one-valued function of two variables $z = f(x,y)$. Since the function is one-valued, any line parallel to z axis cuts this surface exactly at one point.

Definition 1. A smooth surface S is said to be *two-sided*, if a normal vector is moved along any closed curve on the surface so that upon return to the starting point the direction of the normal is the same as it was originally. In the opposite case the surface is called one-sided.

A well known example of the one-sided surface is the *Möbius band.* It consists of a strip of paper with ends joined together to form a loop, but with one end given a half twist before the ends are joined.

For a two-sided surface we differ the upper and the lower side of the surface. The upper side of the surface is the side, where the normal vector forms an acute angle with z axis. The lower side of the surface is the side, where the normal vector forms an obtuse angle with z axis.

The evaluation of the surface integral over the projection depends on the side of the surface over which we integrate. If the function $Z(x, y, z)$ is continuous at any point of the smooth surface $z = f(x, y)$, then the surface integral over the projection onto xy plane is computed by the formula.

$$\iint\limits_S Z(x, y, z)dxdy = \pm \iint\limits_D Z(x, y, f(x, y))dxdy \tag{3.50}$$

On the right side of this formula is a standard double integral, where D denotes the projection of the surface S onto xy plane. Using this formula, we choose the sign "$+$", if we integrate over the upper side of surface and we choose the sign "$-$", if we integrate over the lower side of the surface. So, for any problem there has to be said over which side of the surface we need to integrate.

If the function $Y(x, y, z)$ is continuous at any point of the smooth surface $y = g(x, z)$, then the surface integral over the projection onto xz plane is computed by the formula

$$\iint\limits_S Y(x, y, z)dxdz = \pm \iint\limits_{D'} Y(x, g(x, z), z)dxdz \tag{3.51}$$

In this formula D' denotes the projection of S onto xz plane and the choice of the sign $+$ or $-$ depends on over which side of the surface the integration is carried out (i.e. does the normal of the surface forms with y axis acute or obtuse angle).

If the function $X(x, y, z)$ is continuous at any point of the smooth surface $x = h(y, z)$, then the surface integral over the projection onto yz plane is computed by the formula

$$\iint\limits_S X(x, y, z)dydz = \pm \iint\limits_{D''} X(h(y, z), y, z)dydz \tag{3.52}$$

Here D'' denotes the projection of S onto yz plane and the choice of the sign $+$ or $-$ depends on over which side of the surface the integration is carried out (i.e. does the normal of the surface forms with x axis acute or obtuse angle).

Example. Compute the surface integral

$$\iint\limits_S z^2 dxdy$$

where S is the upper side of the portion of cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 0$ and $z = 1$.

This portion of cone is sketched in Figure 8.8. The projection D onto xy plane of this portion of cone is the disk $x^2 + y^2 \leq 1$. Hence by (3.50)

$$\iint_{\sigma} z^2 dxdy = \iint_{D} (x^2 + y^2) dxdy$$

Since the region of integration is the disk, we convert the double integral into polar coordinates. For this disk $0 \leq \varphi \leq 2\pi$ and $0 \leq \rho \leq 1$, thus,

$$\iint_{D} (x^2 + y^2) dxdy = \int_{0}^{2\pi} d\varphi \int_{0}^{1} \rho^2 \cdot \rho d\rho$$

Now we compute

$$\int_{0}^{1} \rho^3 d\rho = \left. \frac{\rho^4}{4} \right|_{0}^{1} = \frac{1}{4}$$

and

$$\int_{0}^{2\pi} \frac{1}{4} d\varphi = \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$$

4 Series

4.1 Series. Sum of series

The series is an infinite sum

$$u_1 + u_2 + \ \ldots \ + u_k + \ \ldots \ = \sum_{k=1}^{\infty} u_k \tag{4.53}$$

The addends in this infinite sum are called the terms of the series and u_k is called the *general term*. If we assign to k some natural number, we get the related term of the series. In (4.53) the k is called the index of summation and note that the letter we use to represent the index can be any integer variable i, j, l, m, n, ... The first index is 1 for convenience, actually it can be any integer. We can write (4.53) as

$$\sum_{k=1}^{\infty} u_k = \sum_{k=0}^{\infty} u_{k+1} = \sum_{k=2}^{\infty} u_{k-1} = \ldots$$

A *number series* is the series, whose terms are numbers. In our course we consider the series of real numbers. A *functional series* is the series, whose terms are functions of the variable x, i.e. $u_k = u_k(x)$, $k = 1,\ 2,\ \ldots$.

A geometric series is the series

$$a + aq + aq^2 + \ \ldots \ + aq^k + \ \ldots \ = \sum_{k=0}^{\infty} aq^k \tag{4.54}$$

where each successive term is produced by multiplying the previous term by a constant number q (called the *common ratio* in this context).

The harmonic series is the series

$$1 + \frac{1}{2} + \frac{1}{3} + \ \ldots \ + \frac{1}{k} + \ \ldots \ = \sum_{k=1}^{\infty} \frac{1}{k} \tag{4.55}$$

The sum of the first n terms

$$S_n = \sum_{k=1}^{n} u_k$$

is called the nth partial sum of the series. The partial sums

$$S_1 = u_1$$

$$S_2 = u_1 + u_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$S_n = u_1 + u_2 + \ \ldots \ + u_n$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

define the sequence of partial sums

$$S_1,\ S_2,\ \ldots,\ S_n,\ \ldots \tag{4.56}$$

Definition. A series (4.53) is said to *converge* or to *be convergent* when the sequence (4.56) of partial sums has a finite limit. If the limit of (4.56) is infinite or does not exist, the series is said to *diverge* or to *be divergent.* When the limit of partial sums

$$\lim_{n\to\infty} S_n = S$$

exists, it is called the *sum of the series* and one writes

$$S = \sum_{k=1}^{\infty} u_k$$

It is important not to get sequences and series confused! A sequence is a list of numbers written in a specific order while an infinite series is a limit of a sequence and hence, if it exists will be a single value.

Example 1. The sum of the first n terms, i.e. the $n-1$st partial sum of the geometric series is

$$S_{n-1} = \sum_{k=0}^{n-1} aq^k = \frac{a(1-q^n)}{1-q}$$

If $|q| < 1$, then

$$\lim_{n\to\infty} q^n = 0$$

thus,

$$\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} \frac{a(1-q^n)}{1-q} = \lim_{n\to\infty} \frac{a}{1-q} - \lim_{n\to\infty} \frac{aq^n}{1-q} = \frac{a}{1-q}$$

So, if $|q| < 1$, then the geometric series converges and the sum is

$$S = \frac{a}{1-q}$$

If $q > 1$, then

$$\lim_{n\to\infty} q^n = \infty$$

therefore,

$$\lim_{n\to\infty} S_{n-1} = \infty$$

and the geometric series is divergent If $q < -1$, then $\lim\limits_{n\to\infty} q^n$ does not exist and hence, $\lim\limits_{n\to\infty} S_{n-1}$ does not exist and the geometric series is divergent. If $q = 1$, then the $n-1$st partial sum

$$S_n = \sum_{k=0}^{n-1} aq^k = \sum_{k=0}^{n-1} a = na$$

and the limit $\lim\limits_{n\to\infty} S_{n-1} = \lim\limits_{n\to\infty} = na = \infty$. If $q = -1$, then the $S_0 = a$, $S_1 = a - a = 0$, $S_2 = a - a + a = a$, $S_3 = a - a + a - a = 0$, …We obtain the sequence of partial sums

$$a,\ 0,\ a,\ 0,\ \ldots$$

which has no limit. Therefore, for $q = \pm 1$ the geometric series is divergent.

Conclusion. If $|q| < 1$, then the geometric series (4.54) converges and if $|q| \geq 1$ then the geometric series diverges.

Example 2. To find the nth partial sum S_n of the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

we use the partial fractions decomposition

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

We obtain

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)}$$
$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

The limit of this sequence, i.e. the sum of this series

$$S = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

If we ignore the first term the remaining terms will also be a series that will start at $k = 2$ instead of $k = 1$ So, we can rewrite the original series (4.53) as follows,

$$\sum_{k=1}^{\infty} u_k = u_1 + \sum_{k=2}^{\infty} u_k$$

We say that we've stripped out the first term. We could have stripped out the first two terms

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \sum_{k=3}^{\infty} u_k$$

and first any number of terms respectively,

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \ldots + u_m + \sum_{k=m+1}^{\infty} u_k = \sum_{k=1}^{m} u_k + \sum_{k=m+1}^{\infty} u_k$$

The first sum on the right side of this equality is the mth partial

$$\sum_{k=1}^{m} u_k$$

sum of series (4.53). This is a finite sum, which is always finite. Assuming that $n > m$, we can write the nth partial sum

$$\sum_{k=1}^{n} u_k = \sum_{k=1}^{m} u_k + \sum_{k=m+1}^{n} u_k$$

or

$$S_n = S_m + S_{n-m}$$

where

$$S_{n-m} = \sum_{k=m+1}^{n} u_k$$

Now, if S_n has the finite limit as $n \to \infty$, then S_{n-m} must have also the finite limit. Conversely, if S_{n-m} has the finite limit as $n \to \infty$, then adding the finite sum S_m leaves the limit finite.

Similarly, S_n has the infinite limit or does not have the limit if and only if S_{n-m} has also the infinite limit or has no limit.

Conclusion. Stripping out the finite number of terms from the beginning of the series leaves the convergent series convergent and divergent series divergent. As well, adding the finite number of terms to the beginning of the series does not make the convergent series divergent and does not make the divergent series convergent.

4.2 Necessary condition for convergence of series

Suppose that the series (4.53) converges to the sum S, i.e.

$$\lim_{n\to\infty} S_n = S$$

The nth partial sum can be written

$$S_n = \sum_{k=1}^{n} u_k = \sum_{k=1}^{n-1} u_k + u_n$$

or

$$S_n = S_{n-1} + u_n$$

hence,

$$u_n = S_n - S_{n-1}$$

The convergence of the series gives, since if $n \to \infty$ then $n-1 \to \infty$,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

We have proved an essential theorem, so called *necessary condition for the convergence of the series.*

Theorem 1. If the series (4.53) converges, then the limit of the general term

$$\lim_{n\to\infty} u_n = 0 \tag{4.57}$$

This theorem gives us a requirement for convergence but not a guarantee of convergence. In other words, the converse is not true. If $\lim\limits_{n\to\infty} u_n = 0$ the series may actually diverge. For example, the limit of the general term of the harmonic series (4.55)

$$\lim_{k\to\infty} \frac{1}{k} = 0$$

but the harmonic series is divergent. It will be a couple of subsections before we can prove this, so at this point the reader has just to believe this and know that it's possible to prove the divergence.

In order for a series to converge the series terms must go to zero in the limit. If the series terms do not go to zero in the limit then there is no way the series can converge since this would contradict the theorem, i.e. there holds.

Conclusion (the divergence test). If $\lim\limits_{n\to\infty} u_n \neq 0$ then the series (4.53) diverges.

For example the series

$$\sum_{k=1}^{\infty} 1$$

is divergent because the limit of the constant term is that constant,

$$\lim_{k\to\infty} 1 = 1 \neq 0$$

4.3 Convergence tests of positive series

In Mathematical analysis there exist a lot of tests that give us the possibility to decide whether the series converges or diverges. In this subsection we are going to consider the positive series, i.e. the series (4.53), whose all terms are positive:

$$u_k \geq 0, \quad k = 1,\ 2,\ \ldots$$

4.3.1 Comparison test

The nth partial sum of the series (4.53) is

$$S_n = S_{n-1} + u_n$$

Since for any index n $u_n \geq 0$, then

$$S_n \geq S_{n-1}$$

that means, the sequence of partial sums of the positive series is monotonically increasing. We had the theorem in Mathematical analysis I, which stated that any bounded monotonically increasing sequence has the finite limit. So, if we have succeeded to prove that the sequence of the partial sums of the positive series is bounded, we have proved the existence of the finite limit of the sequence of partial sums, that is, we have proved the convergence of the positive series.

The sequence

$$S_1, \ S_2, \ \ldots, \ S_n, \ \ldots$$

has the finite limit means by the definition of the limit that for any $\varepsilon > 0$ there exists the index $N > 0$ such that for all $n \geq N$

$$|S_n - S| < \varepsilon$$

This inequality is identical to the inequalities

$$-\varepsilon < S_n - S < \varepsilon$$

or

$$S - \varepsilon < S_n < S + \varepsilon$$

which means the sequence is bounded. We have proved the following theorem.

Theorem 1. The positive series (4.53) is convergent if and only if the sequence of its partial sums is bounded.

Suppose that we have another positive series

$$\sum_{k=1}^{\infty} v_k \tag{4.58}$$

and we know whether it converges or diverges. For instance we know that the geometric series (4.54) converges if $|q| < 1$ and diverges if $|q| \geq 1$. We know that the harmonic series is divergent and we know that

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

is convergent.

Theorem 2 (the comparison test). 1) If for any $k = 1, \ 2, \ 3, \ \ldots$

$$u_k \leq v_k$$

then the convergence of the series (4.58) yields the convergence of the series (4.53).

2) If for any $k = 1,\ 2,\ 3,\ \ldots$

$$u_k \geq v_k$$

then the divergence of the series (4.58) yields the divergence of the series (4.53).

Example 1. Prove that the series

$$1 + \frac{1}{4} + \frac{1}{9} + \ \ldots \ + \frac{1}{k^2} + \ \ldots \ = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges.

We know that the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=2}^{\infty} \frac{1}{(k-1)k}$$

converges. For any $k = 2,\ 3,\ \ldots$ it is obvious that

$$\frac{1}{k^2} < \frac{1}{(k-1)k}$$

and by Theorem 2 the series

$$\sum_{k=2}^{\infty} \frac{1}{k^2}$$

converges. Adding the term 1 to the beginning of the series preserves the convergence.

Example 2. Prove that the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ \ldots \ + \frac{1}{\sqrt{k}} + \ \ldots \ = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

diverges.

For any $k \geq 1$ there holds the inequality $\sqrt{k} \leq k$ hence,

$$\frac{1}{\sqrt{k}} > \frac{1}{k}$$

The harmonic series (4.55) diverges thus, by Theorem 2 the series given diverges also.

4.3.2 D'Alembert's test

Sometimes the D'Alembert's test is referred as the ratio test. We consider again the positive series (4.53).

Theorem (D'Alembert's test). Suppose there exists the limit

$$\lim_{k\to\infty} \frac{u_{k+1}}{u_k} = D$$

1) If $D < 1$, then the series (4.53) converges.

2) If $D > 1$, then series (4.53) diverges.

3) If $D = 1$, then this test us inconclusive, because there exist both convergent and divergent series that satisfy this case.

Example 1. Does the series $\sum\limits_{k=1}^{\infty} \dfrac{1}{k!}$ converge or diverge?

The ratio of two consecutive terms $u_{k+1} = \dfrac{1}{(k+1)!}$ and $u_k = \dfrac{1}{k!}$ is

$$\frac{u_{k+1}}{u_k} = \frac{\dfrac{1}{(k+1)!}}{\dfrac{1}{k!}} = \frac{k!}{(k+1)k!} = \frac{1}{k+1}$$

and the limit of this ratio

$$D = \lim_{k\to\infty} \frac{1}{k+1} = 0$$

Since $D = 0$, this series converges by the D'Alembert's test.

Example 2. Does the series $\sum\limits_{k=1}^{\infty} \dfrac{1}{k^2}$ converge or diverge?

Compute the limit

$$D = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{\dfrac{1}{(k+1)^2}}{\dfrac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} = 1$$

Since $D = 1$, the D'Alembert's test is inconclusive, but we know that by the comparison test that this series converges.

Example 3. Does the series $\sum\limits_{k=1}^{\infty} \dfrac{1}{k}$ converge or diverge?

For the harmonic series we have

$$D = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{\dfrac{1}{k+1}}{\dfrac{1}{k}} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

so, the harmonic series cannot be handled by the D'Alembert's test, but we know that the series diverges.

4.3.3 Cauchy test

Cauchy test is also known as root test of convergence of a series. Let us consider the positive series (4.53) again.

Theorem (Cauchy test). Suppose there exists the limit

$$\lim_{k\to\infty} \sqrt[k]{u_k} = C$$

1) If $C < 1$, then the series (4.53) converges.
2) If $C > 1$, then series (4.53) diverges.
3) If $C = 1$, then this test us inconclusive.

Example 1. Determine if the series

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k}$$

is convergent or divergent?

To use the Cauchy test we find $\sqrt[k]{u_k} = \dfrac{\sqrt[k]{k^2}}{2}$ and evaluate the limit

$$C = \lim_{k\to\infty} \sqrt[k]{u_k} = \lim_{k\to\infty} \frac{\sqrt[k]{k^2}}{2} = \frac{1}{2} \lim_{k\to\infty} k^{\frac{2}{k}}$$

Since we have the indeterminate form ∞^0, we apply the L'Hospital's rule

$$\lim_{k\to\infty} \ln k^{\frac{2}{k}} = \lim_{k\to\infty} \frac{2}{k} \ln k$$
$$= \lim_{k\to\infty} \frac{(2\ln k)'}{k'} = \lim_{k\to\infty} \frac{2}{k} = 0$$

and

$$C = \frac{1}{2} e^0 = \frac{1}{2} < 1$$

So, by the Cauchy test the series is convergent.

Example 2. Determine if the series

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{k^2}$$

is convergent or divergent?

The kth root of the general term is

$$\sqrt[k]{u_k}=\sqrt[k]{\left(1+\frac{1}{k}\right)^{k^2}}=\left(1+\frac{1}{k}\right)^{k}$$

and the limit

$$C=\lim_{k\to\infty}\sqrt[k]{u_k}=\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k}=e>1$$

Hence, by the Cauchy test the series is divergent.

4.3.4 Integral test

Let us consider the a positive series (4.53) once more.

Theorem 5 (Integral test). Suppose $u(x)$ is a continuous positive decreasing on interval $[1;\infty)$ function, whose values for the integer arguments are the terms of series (4.53), i.e. $u(k)=u_k$. Then

1) if the improper integral (4.53) $\int\limits_1^{\infty} u(x)dx$ is convergent so is the series (4.53);

1) if the improper integral (4.53) $\int\limits_1^{\infty} u(x)dx$ is divergent so is the series (4.53).

Example 4. Prove that the harmonic series

$$\sum_{k=1}^{\infty}\frac{1}{k}$$

diverges.

To apply the integral test we define the decreasing function $u(x)=\dfrac{1}{x}$, whose values for the integer arguments $x=k$ are

$$u_k=u(k)=\frac{1}{k}$$

The improper integral is divergent because

$$\int_1^\infty \frac{dx}{x} = \lim_{N\to\infty} \int_1^N \frac{dx}{x} = \lim_{N\to\infty} \ln|x| \Big|_1^N = \lim_{N\to\infty} \ln N = \infty$$

By the Integral test the harmonic series is divergent.

4.4 Alternating series. Leibnitz's test.

The last tests that we looked at for series convergence have required that all the terms in the series be positive. The test that we are going to look into in this subsection will be a test for alternating series. An alternating series is any series

$$u_1 - u_2 + u_3 - u_4 + \ \ldots = \sum_{k=1}^{\infty} (-1)^{k+1} u_k \tag{4.59}$$

or

$$-u_1 + u_2 - u_3 + u_4 - \ \ldots = \sum_{k=1}^{\infty} (-1)^{k} u_k$$

where $u_k > 0, \ \ k = 1, \ 2, \ \ldots$

The second alternating series we can write

$$\sum_{k=1}^{\infty} (-1)^k u_k = -\sum_{k=1}^{\infty} (-1)^{k+1} u_k$$

therefore, it's enough to look at for convergence of the series (4.59).

Theorem 1. (Leibnitz's test) If

1) $u_k > u_{k+1}$, $k = 1, \ 2, \ \ldots$ and

2) $\lim\limits_{k\to\infty} u_k = 0$, then the alternating series (4.59) converges.

Example. For the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$

both of the assumptions of the theorem hold because

$$1 > \frac{1}{2} > \ldots > \frac{1}{k} > \frac{1}{k+1} > \ldots$$

and

$$\lim_{k\to\infty} \frac{1}{k} = 0$$

Hence, this series is convergent.

4.5 Absolute and conditional convergence

In this subsection we assume that the terms of the series

$$\sum_{k=1}^{\infty} u_k \tag{4.60}$$

can have whatever signs.

Definition 1. The series (4.60) is called *absolutely convergent* if the series

$$|u_1| + |u_2| + |u_3| + \ldots = \sum_{k=1}^{\infty} |u_k|$$

is convergent.

Theorem 1. If the series (4.60) is absolutely convergent then it is also convergent.

Proof. The definition of the absolute value

$$|u_k| = \begin{cases} u_k, & \text{if } u_k \geq 0 \\ -u_k, & \text{if } u_k < 0 \end{cases}$$

gives us that

$$0 \leq u_k + |u_k| \leq 2|u_k|$$

Since we are assuming that

$$\sum_{k=1}^{\infty} |u_k|$$

is convergent then

$$\sum_{k=1}^{\infty} 2|u_k| = 2\sum_{k=1}^{\infty} |u_k|$$

is also convergent because 2 times finite value will still be finite. The comparison test gives us that

$$\sum_{k=1}^{\infty} (u_k + |u_k|)$$

is also a convergent series. Now the series (4.60)

$$\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} (u_k + |u_k| - |u_k|) = \sum_{k=1}^{\infty} (u_k + |u_k|) - \sum_{k=1}^{\infty} |u_k|$$

is the difference of two convergent series, i.e. convergent.

By Theorem 1 series that are absolutely convergent are guaranteed to be convergent. However, series that are convergent may or may not be absolutely convergent.

Definition 2. The series (4.60) which is convergent but not absolutely convergent is called *conditionally convergent*.

Example 1. Alternating harmonic series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$$

is convergent by Leibnitz's test, but the series

$$\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{1}{k}\right| = \sum_{k=1}^{\infty}\frac{1}{k}$$

is the harmonic series. By Integral test the harmonic series diverges hence, alternating harmonic series is a conditionally convergent series.

Example 2. Determine if the series $\sum\limits_{k=1}^{\infty}\frac{\sin k}{k^2}$ is absolutely convergent, conditionally convergent or divergent.

Notice that this is not an alternating series. Since $|\sin k| \leq 1$ for any integer k, then

$$\left|\frac{\sin k}{k^2}\right| \leq \frac{1}{k^2}$$

We know that the series $\sum\limits_{k=1}^{\infty}\frac{1}{k^2}$ converges hence, by Comparison test the series

$$\sum_{k=1}^{\infty}\left|\frac{\sin k}{k^2}\right|$$

converges, i.e. the series $\sum\limits_{k=1}^{\infty}\frac{\sin k}{k^2}$ is absolutely convergent and Theorem 1 guarantees its convergence.

While the convergence of the positive series takes place because of the terms are decreasing with the sufficient speed, then the conditional convergence happens because the terms reduce each other.

4.6 Power series

A series of functions is the series, whose terms are the functions of some variable, suppose x

$$\sum_{k=1}^{\infty} u_k(x) \tag{4.61}$$

If we assign to the variable x a certain value x_0 that is in domains of all u_k and substitute it into all these functions, we have the numerical values $u_k(x_0)$, i.e for $x = x_0$ the series (4.61) is a number series.

Example. Let's examine the series of functions

$$1 + x + x^2 + \ldots + x^k + \ldots = \sum_{k=0}^{\infty} x^k \tag{4.62}$$

If the variable x has the value $x = \dfrac{1}{2}$, we get the geometric series

$$\sum_{k=0}^{\infty} \frac{1}{2^k}$$

which is convergent, because the common ratio is $\dfrac{1}{2}$.

Assigning to the variable x the value $x = 1$, we get the number series

$$1 + 1 + 1 + \ldots$$

which diverges by Divergence test. Assigning to the variable x the value $x = -1$, we get the divergent number series

$$1 - 1 + 1 - \ldots + (-1)^k + \ldots$$

Assigning to the variable x the some value $x_0 > 1$, we obtain the number series with general term

$$u_k(x_0) = x_0^k$$

which diverges by Divergence test because

$$\lim_{k\to\infty} x_0^k = \infty$$

Assigning to the variable x the some value $x_0 < -1$, we obtain the number series which diverges by Divergence test because the general term has no limit.

It has turned out that for some values of the variable x the series of functions converges and for other values it diverges.

The partial sums of the series of functions (4.61)

$$S_n(x) = \sum_{k=1}^{n} u_k(x)$$

are also functions of the variable x and define a sequence of functions

$$S_1(x),\ S_2(x),\ \ldots,\ S_n(x),\ \ldots \tag{4.63}$$

Definition. The set X of the values of argument x for which the sequence of partial sums (4.63) is convergent, i.e. there exists the (finite) limit

$$S(x) = \lim_{n\to\infty} S_n(x) \tag{4.64}$$

is called the *region of convergence* of the series of functions (4.61).

It is said that $S(x)$ is the sum of the series (4.61) and one writes

$$S(x) = \sum_{k=1}^{\infty} u_k(x)$$

Power series is a series of power functions

$$\sum_{k=0}^{\infty} c_k x^k \tag{4.65}$$

or in general

$$\sum_{k=0}^{\infty} c_k (x-a)^k \tag{4.66}$$

where the numbers c_k are called the coefficients of the series.

The examination of the properties of those series is very similar therefore, we restrict ourselves with series (4.65).

Example 1. The series

$$1 + x + x^2 + \ \ldots\ + x^k + \ \ldots = \sum_{k=0}^{\infty} x^k$$

is a geometric series for any value of x. This series converges if $|x| < 1$. Hence, the region of convergence of this series is open interval $X = (-1; 1)$ and the sum of this series in this interval is

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \tag{4.67}$$

It turns out that the regions of convergence of power series have such a simple structure.

Theorem 1 (Abel's theorem). If the power series (4.65) converges for some value of x_0, then this series converges absolutely for any value of $|x| < |x_0|$.

Conversely, if the power series (4.65) diverges for some value of x_0, then this series diverges for any value of $|x| > |x_0|$.

According to Abel's theorem there exists a real number R such that for $|x| < R$ the series (4.65) converges and for $|x| > R$ diverges. This real number R is called the *radius of convergence* of the series (4.65) and the interval $(-R; R)$ the *interval of convergence* of this series.

Remark. At the endpoints $x = R$ and $x = -R$ of the interval of convergence the series (4.65) may converge and may diverge. Therefore, to completely identify the interval of convergence all that we have to do is determine if the power series will converge for $x = R$ or $x = -R$. If the power series converges for one or both of these values then we'll need to include those in the interval of convergence.

There are a lot of possibilities to determine the radius of convergence of power series (4.65). One of these possibilities is to use the formula.

$$R = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right| \tag{4.68}$$

Example. Find the intervals of convergence of power series

$$\sum_{k=1}^{\infty} x^k$$

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$

and

$$\sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

The radius of convergence is 1 for all of three series. The coefficient of the first series are $c_k = 1$ hence,

$$R = \lim_{k\to\infty} \frac{1}{1} = 1$$

The coefficients of the second series are $c_k = \dfrac{1}{k}$ and

$$R = \lim_{k\to\infty} \frac{k+1}{k} = 1$$

The coefficients of the third series are $c_k = \dfrac{1}{k^2}$ and

$$R = \lim_{k\to\infty} \frac{(k+1)^2}{k^2} = 1$$

thus, all three series are convergent if $-1 < x < 1$ and diverges if $|x| > 1$. Determine if these series will converge for $x = 1$ or $x = -1$.

The general term of the first series at the right endpoint is $1^k = 1$, whose limit $1 \neq 0$ hence, the series diverges. At the left endpoint the general term is $(-1)^k$, which has no limit as $k \to \infty$, i.e. the series diverges again and the interval of convergence of the first series is $(-1; 1)$

The general term of the second series at the right endpoint is $\dfrac{1}{k}$ hence, the second series is at the right endpoint the harmonic series, which is divergent. At the left endpoint the general term is $\dfrac{(-1)^k}{k}$, i.e. the second series is at the left endpoint the alternating harmonic series, which converges by Leibnitz's test. Thus, the interval of convergence of the second series is $[-1; 1)$.

The general term of the second series at the right endpoint is $\dfrac{1}{k^2}$ and at the left endpoint $\dfrac{(-1)^k}{k^2}$. The absolute value of both of these is $\dfrac{1}{k^2}$. By Example 1 of subsection 8.3 the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges thus, the third series converges at both endpoints and the interval of convergence is $[-1; 1]$.

Inside the interval of convergence of power series it's possible to prove.

Conclusion 1. If the radius of convergence of the power series (4.65) is R, then the sum of this series is continuous on any interval $[a; b] \subset (-R; R)$.

Conclusion 2. If the radius of convergence of the power series (4.65) is R, then this series can be integrated term by term on any interval $[a; b] \subset (-R; R)$.

Conclusion 3. If the radius of convergence of the power series (4.65) is R, then this series can be differentiated term by term on any interval $[a; b] \subset (-R; R)$.

Now, using the sum of the geometric series (4.67) and conclusions 2 and 3, we can find the power series expansions for many functions.

Example 1. Multiplying both sides of (4.67) by x gives

$$\frac{x}{1-x} = x \cdot \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} x^{k+1}$$

and the radius of convergence is still 1. It's easy to verify that

$$\left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}$$

and using the term by term differentiation we get the power series expansion of this derivative

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(x^{k+1})' = \sum_{k=0}^{\infty}(k+1)x^k$$

and the radius of convergence of the series obtained is 1 again.

Example 2. If we substitute in (4.67) the variable x by $-x^2$, we get

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty}(-x^2)^k = \sum_{k=0}^{\infty}(-1)^k x^{2k}$$

and this series converges if $|-x^2| < 1$, which is equivalent to $|x| < 1$.

Since

$$\arctan x = \int\limits_0^x \frac{dx}{1+x^2}$$

, we obtain the power series of arc tangent function integrating the last series term by term in limits from 0 to x provided $|x| < 1$.

$$\arctan x = \sum_{k=0}^{\infty}(-1)^k \int\limits_0^x x^{2k}dx = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$$

and the radius of convergence is 1 hence, the interval of convergence is $(-1;1)$.

At the left endpoint of the interval of convergence we get the series

$$\sum_{k=0}^{\infty}(-1)^k \frac{(-1)^{2k+1}}{2k+1} = -\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

and at the right endpoint

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

Both series obtained are the alternating series, which converge by the Leibnitz's test and therefore, the interval of convergence of the series obtained is $[-1;1]$.

So, it may happen that the series obtained as the result of term by term integration converges at one or both of the endpoints, despite of the initial series diverges at the endpoints.

4.7 Taylor's and Maclaurin's series

Suppose that the function $f(x)$ is differentiable infinitely many times in the neighborhood of a. If the coefficients c_k of the power series

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

are computed by the formula

$$c_k = \frac{f^{(k)}(a)}{k!} \tag{4.69}$$

then these coefficients are called Taylor's coefficients and the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{4.70}$$

is called Taylor's series of the function $f(x)$ in the neighborhood of a or Taylor's series of the function $f(x)$ in powers $x-a$. The nth partial sum of this series is the Taylor's polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

By Taylor's formula the function $f(x)$ can be represented as

$$f(x) = P_n(x) + R_n(x)$$

that is the sum of the Taylor's polynomial and the remainder.

We know that Lagrange form of the remainder of the Taylor's formula is

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(a + \Theta(x-a))$$

where $0 < \Theta < 1$

If

$$\lim_{n\to\infty} R_n(x) = 0$$

then

$$\lim_{n\to\infty} P_n(x) = f(x)$$

which means that the sequence of partial sums of Taylor's series converges to the function $f(x)$.

Therefore, the series (4.72) represents the function $f(x)$ if and only if the limit of the remainder equals to 0. If $\lim\limits_{n\to\infty} R_n(x) \neq 0$, then the Taylor's series of the function $f(x)$ may still converge but it does not represent the function $f(x)$.

Taylor's series in the neighborhood of $a = 0$, i.e. Taylor's series in powers x

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \tag{4.71}$$

is called Maclaurin's series.

4.8 Maclaurin's series of functions e^x, $\sin x$ and $\cos x$

In Mathematical analysis I we have proved that Maclaurin's formula of nth degree of the exponential function e^x is

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{n!}x^n + R_n(x)$$

and that the limit of the remainder

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} e^{\Theta x} = 0$$

for each $x \in \mathbb{R}$ and for $0 < \theta < 1$. Consequently, Maclaurin's series represents the function e^x for every real x, i.e.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Also it has been proved that Maclaurin's formula of $2n+1$st degree of the sine function $\sin x$ is

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n+1}(x)$$

whose remainder is

$$R_{2n+1}(x) = \frac{x^{2n+2}}{(2n+2)!} \sin\left(\Theta x + (n+1)\pi\right)$$

Since for every $x \in \mathbb{R}$ and for $0 < \theta < 1$

$$\lim_{n\to\infty} R_{2n+1}(x) = 0$$

Maclaurin's series represents the function $\sin x$ for every real x:

$$\sin x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$$

As well it has been proved that Maclaurin's formula of $2n$th degree of the cosine function $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n}(x)$$

and the remainder

$$R_{2n}(x) = \frac{x^{2n+1}}{(2n+1)!} \cos\left(\Theta x + (2n+1)\frac{\pi}{2}\right)$$

Again, for every $x \in \mathbb{R}$ and for $0 < \theta < 1$

$$\lim_{n\to\infty} R_{2n}(x) = 0$$

hence, Maclaurin's series represents the function $\cos x$ for every real x:

$$\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots$$

4.9 Fourier series of 2π-periodic functions

For an infinitely many times differentiable function $f(x)$ Maclaurin's series expansion is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \tag{4.72}$$

Here we have expanded the function $f(x)$ with respect to *system of power fuctions*

$$\{1,\ x, x^2,\ \ldots\}$$

Another system of functions is the *system of trigonometric functions*

$$\{1;\ \sin x;\ \cos x;\ \sin 2x;\ \cos 2x;\ \ldots;\ \sin kx;\ \cos kx;\ \ldots\} \tag{4.73}$$

The series with respect to system of trigonometric functions

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{4.74}$$

is called *trigonometric series*. We shall see later that taking the constant term as $\frac{a_0}{2}$ rather that a_0 is a convenience that enables us to make a_0 fit a general result.

Suppose the function $f(x)$ is 2π-periodic i.e. for each $x, x+2\pi \in X$

$$f(x+2\pi) = f(x)$$

which means that the values of the function are repeated at interval 2π in its domain. For this 2π-periodic function we find coefficients of trigonometric series (4.74)

$$a_0 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)dx \tag{4.75}$$

$$a_k = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos kx dx \quad k = 1,\ 2,\ \ldots \tag{4.76}$$

and

$$b_k = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin kx dx \quad k = 1,\ 2,\ \ldots \tag{4.77}$$

The coefficients a_0, a_k and b_k defined by (4.75), (4.76) and (4.77), respectively, are called the *Fourier coefficients* of the function $f(x)$ and the trigonometric series with these coefficients is called the *Fourier series* of the function $f(x)$.

We have got the formulas to compute the Fourier coefficients. But if we compute the Fourier coefficients by the formulas (4.75), (4.76) and (4.77) and write the Fourier series expansion

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

we don't know whether this expansion converges and if it converges, converges it to $f(x)$ or to some other value. For now we are just saying that associated with the function $f(x)$ on $[-\pi;\pi]$ is a certain series called Fourier series. Therefore we write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \tag{4.78}$$

The equality sign $=$ can be written instead of $\sim$ only if we have proved the convergence of the Fourier series to the function $f(x)$.

Example 1. Find the Fourier coefficients and Fourier series of the square-wave function defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad -\pi < x \leq 0 \\ 1 & \text{if} \quad 0 < x \leq \pi \end{cases} \qquad \text{and} \quad f(x + 2\pi) = f(x)$$

So $f(x)$ is periodic with period 2π. Using the formulas (4.75), (4.76) and

(4.77), we find the Fourier coefficients

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{0}^{\pi} dx = \frac{1}{\pi} \cdot \pi = 1$$

$$a_k = \frac{1}{\pi} \int_{0}^{\pi} \cos kx dx = \frac{1}{k\pi} \sin kx \Big|_{0}^{\pi} = 0$$

and

$$b_k = \frac{1}{\pi} \int_{0}^{\pi} \sin kx dx = -\frac{1}{k\pi} \cos kx \Big|_{0}^{\pi} = -\frac{1}{k\pi}((-1)^k - 1) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{2}{k\pi} & \text{if } k \text{ is odd} \end{cases}$$

Thus, $a_k = 0$ and and $b_{2k} = 0$ for every $k = 1, 2, \ldots$. Fourier series of square-wave function is

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \frac{2}{5\pi} \sin 5x + \ldots$$

or

$$f(x) \sim \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin(2k+1)x$$

The following theorem gives a sufficient condition for convergence of the Fourier series.

Theorem (Dirichlet's theorem). If $f(x)$ is a bounded 2π-periodic function which in any one period has at most a finite number of local maxima and minima and a finite number of points of jump discontinuity, then the Fourier series of $f(x)$ converges to $f(x)$ at all points where $f(x)$ is continuous and converges to the average of the right- and left-hand limits of $f(x)$ at each point where $f(x)$ is discontinuous.

The square-wave function has on half-open interval $(-\pi;\pi]$ one local maximum equal to 1 and one local minimum equal to 0 and two points of jump discontinuity 0 an π. Hence, at any point in the open intervals $(-\pi;0)$ and $(0;\pi)$ Fourier series converges to $f(x)$. The left-hand limit at 0 is $f(0-) = \lim\limits_{x\to 0-} f(x) = 0$ and the right-hand limit at 0 is $f(0+) = \lim\limits_{x\to 0+} f(x) = 1$ and the average of these one-sided limits is $\dfrac{0+1}{2} = \dfrac{1}{2}$. The left-hand limit at π is $f(\pi-) = \lim\limits_{x\to\pi-} f(x) = 1$ and the right-hand limit at π is $f(\pi+) = \lim\limits_{x\to\pi+} f(x) = 0$ and the average of one-sided limits is $\dfrac{1+0}{2} = \dfrac{1}{2}$.

Thus, at the points of discontinuity the Fourier series of the square-wave function converges to $\dfrac{1}{2}$. Since $\sin((2k+1)\cdot 0) = 0$ and $\sin((2k+1)\pi) = 0$ for any integer k, then the direct computation also gives

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin((2k+1)0) = \frac{1}{2}$$

and

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin((2k+1)\pi) = \frac{1}{2}$$