Numerical Methods for Equations of Mathematical Physics. Homework Problems

Each student has to take one problem from the list A one problem from and the list B. Solutions may be presented either electronically or in the printed form.

List A

A1. Solve the problem

$$-3u''(x) + (x+2)u(x) = 4x, \quad x \in (-1,1),$$
$$u'(-1) + 4u(-1) = 3, \quad -u'(1) + 2u(1) = 0$$

by means of the finite difference method with stepsize $h = 0.001$. Plot the graph.

A2. Solve the problem

$$u''(x) - [\cos(x) + 4]u(x) = x + 7, \quad x \in (0,5),$$
$$u'(0) = 0, \quad u(5) = 3$$

by means of the finite difference method with stepsize $h = 0.01$. Plot the graph.

A3. Solve the problem

$$-u''(x) + 2xu(x) \;=\; x^2\,,\; x \in (1,2)\,,$$
$$u'(1) = u(2) = 0,$$

by means of the Galerkin finite element method with shape functions

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere,} \end{cases}$$

where $x_i = 1 + ih$, $h = 0.05$. Plot the graph.

A4. Solve the problem

$$\begin{aligned}-u''(x)+(4-x)u(x) &= x+5\,,\ x\in(0,2)\,,\\ u'(0)=1,\ u'(2)=2,\end{aligned}$$

by means of the Galerkin finite element method with shape functions

$$\varphi_i(x)=\begin{cases}\frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{for } x\in[x_{i-1},x_i]\\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & \text{for } x\in[x_i,x_{i+1}]\\ 0 & \text{elsewhere,}\end{cases}$$

where $x_i = ih$, $h = 0.01$. Plot the graph.

A5. Solve the problem

$$\begin{aligned}-(pu')'(x)+u(x) &= 3\,,\ x\in(0,1)\,,\\ u(0)=-1,\ u(1)=1,\end{aligned}$$

where $p(x) = \begin{cases}2 & \text{if } 0<x<0.5\\ 1 & \text{if } 0.5<x<1\end{cases}$ by means of the Galerkin finite element method with shape functions

$$\varphi_i(x)=\begin{cases}\frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{for } x\in[x_{i-1},x_i]\\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & \text{for } x\in[x_i,x_{i+1}]\\ 0 & \text{elsewhere,}\end{cases}$$

where $x_i = ih$, $h = 0.001$. Plot the graph.

A6. Solve the problem

$$\begin{aligned}-u''(x)+6u(x) &= -\sqrt{x}\,,\ x\in(0,1)\,,\\ u(0)=u(1)=0\end{aligned}$$

by means of finite volume method with approximation $u_h = \sum_{j=0}^{n-1} u_i\varphi_i$,

$$\varphi_i(x)=\begin{cases}\frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{for } x\in[x_{i-1},x_i]\\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & \text{for } x\in[x_i,x_{i+1}]\\ 0 & \text{elsewhere}\end{cases}$$

on the mesh $\mathcal{M} = \bigcup_{j=1}^{n-1} \omega_j$, where $\omega_j = (x_{j-1/2}, x_{j+1/2})$, $j = 1,\ldots,n-1$, $x_j = jh$, $j = 1,\ldots,n$, $h = 0.05$ and $n = 20$. Plot the graph.

A7. Solve the problem

$$-((x^2+1)u')'(x) \; = \; x\,, \;\; x \in (0,1)\,,$$
$$u(0) = 2, \;\; u(1) = 1$$

by means of finite volume method with approximation $u_h = \sum_{j=0}^{n-1} u_i \varphi_i$,

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere} \end{cases}$$

on the mesh $\mathcal{M} = \bigcup_{j=1}^{n-1} \omega_j$, where $\omega_j = (x_{j-1/2}, x_{j+1/2})$, $j = 1, \ldots, n-1$, $x_j = jh$, $j = 1, \ldots, n$, $h = 0.05$ and $n = 20$. Plot the graph.

List B

B1. Solve the problem

$$-\Delta u(x,y) + 7u(x,y) = x^2 - y^2, \;\; (x,y) \in \Omega,$$
$$u(x,y) = \cos y, \;\; (x,y) \in \Gamma,$$

where $\Omega = \{(x,y) \,:\, 1 \le x \le 3, \, 1 \le y \le 4\}$ and Γ is the boundary of Ω, by means of the finite difference method with stepsizes $h_x = h_y = 0.02$. Plot the graph.

B2. Solve the problem

$$-\Delta u(x,y) + 1.3u(x,y) = x^2 - y^2, \;\; (x,y) \in \Omega,$$
$$u(x,y) = xy - 5, \;\; (x,y) \in \Gamma,$$

where $\Omega = \{(x,y) \,:\, -1 \le x \le 0, \, 1 \le y \le 2\}$ and Γ is the boundary of Ω, by means of the finite difference method with stepsizes $h_x = h_y = 0.01$. Plot the graph.

B3. Solve the problem

$$-3u_{xx}(x,y) - 4u_{yy}(x,y) + (x^2 + 2y^2)u(x,y) = 7(x-y),$$
$$x \in (0,2), \;\; y \in (0,1),$$

$$u(x,0) = 5x - 3, \; u(x,1) = -2x - 3, \; u(0,y) = -3, \; -yu_x(2,y) + u(2,y) = 7$$

by means of the finite difference method with stepsizes $h_x = h_y = 0.01$. Plot the graph.

B4. Solve the problem

$$
\begin{aligned}
u_t(x,t) - 4u_{xx}(x,t) &= \cos(x+t)\,,\ x \in (-1,1)\,, t \in (0,2),\\
u(-1,t) = u(1,t) = 2\,,\ t \in (0,2),\\
u(x,0) = 1 + x^2\,,\ x \in [-1,1]
\end{aligned}
$$

by means of Cranck-Nicolson scheme with stepsizes $h = \tau = 0.01$. Plot the graph. (NB! Previous version of this problem contained boundary conditions $u(-1,t) = u(1,t) = 1$ that are inconsistent with the initial condition. But solution with these boundary conditions is also accepted.)

B5. Construct an explicit difference scheme of 2nd order local accuracy for the problem

$$
\begin{aligned}
u_{tt}(x,t) + u_t(x,t) - u_{xx}(x,t) - u_x(x,t) &= x + t\,,\ x \in (0,1)\,, t \in (0,4),\\
u(0,t) = u(1,t) = 0\,,\ t \in (0,4),\\
u(x,0) = u_t(x,0) = 0\,,\ x \in [0,1].
\end{aligned}
$$

and solve the problem with the space stepsize $h = 0.01$ and different time stepsizes τ. Numerically, find maximum τ for maintaining the stability.