Let us consider set of vectors of m real components
$\vec{x} = (x_1, x_2, \ldots, x_n)^T$.
Denote this set by $\mathbb{R}^n$.

Norm of vector $\vec{x} \in \mathbb{R}^n$ is a real number $\|\vec{x}\|$ satisfying the following conditions:

(I) $\|\vec{x}\| \geq 0$

(II) $\vec{x} = \vec{0}$ if and only if $\|\vec{x}\| = 0$

(III) $\|\lambda\vec{x}\| \; = \; |\lambda| \, \|\vec{x}\|$ for all $\lambda \in \mathbb{R}$

(IV) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Distance between $\vec{x}, \vec{y} \in \mathbb{R}^n$ is

$$\|\vec{x} - \vec{y}\|.$$

Examples of norms:

$$\|\vec{x}\|_2 \; = \; \left[x_1^2 + x_2^2 + \ldots + x_n^2\right]^{1/2}$$

$$\|\vec{x}\|_1 \; = \; |x_1| + |x_2| + \ldots + |x_n|$$

$$\|\vec{x}\|_\infty \; = \; \max\{|x_1|; |x_2|; \ldots; |x_n|\}$$

Norms of matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

General definition:

$$\|A\| \;=\; \max_{\vec{x}\in\mathbb{R}^n} \frac{\|A\vec{x}\|}{\|\vec{x}\|}\,.$$

$$\|A\|_2 \;=\; \max_{\vec{x}\in\mathbb{R}^n} \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2}, \quad \|A\|_1 \;=\; \max_{\vec{x}\in\mathbb{R}^n} \frac{\|A\vec{x}\|_1}{\|\vec{x}\|_1},$$

$$\|A\|_\infty \;=\; \max_{\vec{x}\in\mathbb{R}^n} \frac{\|A\vec{x}\|_\infty}{\|\vec{x}\|_\infty}$$

Formulas:

$$\|A\|_1 \;=\; \max_{j=1,\dots,n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_\infty \;=\; \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\|_2 \;=\; \sqrt{\lambda_{\max}},$$

where λ_{max} is biggest eigenvalue of $A^T A$.

Methods to solve linear systems of algebraic equations.

$$
\begin{array}{ccccccc}
a_{11}x_1 & + & a_{12}x_2 & + \ldots + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + \ldots + & a_{2n}x_n & = & b_2 \\
 & & \ldots & & & & \\
a_{n1}x_1 & + & a_{n2}x_2 & + \ldots + & a_{nn}x_n & = & b_n
\end{array}
$$

$$A\vec{x} = \vec{b}$$

$$
A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & \ldots & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, \ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{pmatrix}.
$$

Direct methods

Gaussian elimination

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & \dots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} & b_i \\ & \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{pmatrix}$$

$$\begin{pmatrix} 1 & \beta_{12} & \dots & \beta_{1n} & \theta_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & \dots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} & b_i \\ & \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{pmatrix}$$

$$\begin{pmatrix} 1 & \beta_{12} & \dots & \beta_{1n} & \theta_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & \dots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} & b_i \\ & \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{pmatrix}$$

New matrix:

$$\begin{pmatrix} 1 & \beta_{12} & \beta_{13} & \dots & \beta_{1n} & \theta_1 \\ 0 & a'_{22} & a'_{23} & \dots & a'_{2n} & b'_2 \\ 0 & a'_{32} & a'_{33} & \dots & a'_{3n} & b'_3 \\ & & \dots & & & \\ 0 & a'_{n2} & a'_{n3} & \dots & a'_{nn} & b'_n \end{pmatrix}$$

Here

$$a'_{ik} = a_{ik} - a_{i1}\beta_{1k} \quad (k = 2, \dots, n), \qquad b'_i = b_i - a_{i1}\theta_1\,,$$
$$i = 2, \dots, n.$$

$$\begin{pmatrix} 1 & \beta_{12} & \beta_{13} & \ldots & \beta_{1n} & \theta_1 \\ 0 & 1 & \beta_{23} & \ldots & \beta_{2n} & \theta_2 \\ 0 & 0 & a''_{33} & \ldots & a''_{3n} & b''_3 \\ & & \ldots & & & \\ 0 & 0 & a''_{n3} & \ldots & a''_{nn} & b''_n \end{pmatrix}$$

$$\begin{pmatrix} 1 & \beta_{12} & \beta_{13} & \ldots & \beta_{1n} & \theta_1 \\ 0 & 1 & \beta_{23} & \ldots & \beta_{2n} & \theta_2 \\ 0 & 0 & 1 & \ldots & \beta_{3n} & \theta_3 \\ & & \ldots & & & \\ 0 & 0 & 0 & \ldots & 1 & \theta_n \end{pmatrix}$$

$$\begin{aligned} x_1 + \beta_{12}x_2 + \beta_{13}x_3 + \ldots + \beta_{1n}x_n &= \theta_1 \\ x_2 + \beta_{23}x_3 + \ldots + \beta_{2n}x_n &= \theta_2 \\ \ldots & \\ x_{n-1} + \beta_{n-1,n}x_n &= \theta_{n-1} \\ x_n &= \theta_n. \end{aligned}$$

Band matrices

$$
\begin{pmatrix}
\times & \times & & & & & & \\
\times & \times & \times & & & & & \\
 & \times & \times & \times & & & & \\
 & & \times & \times & \times & & & \\
 & & & & \cdots & & & \\
 & & & & & \times & \times & \times \\
 & & & & & & \times & \times
\end{pmatrix}
$$

$$
\begin{pmatrix}
\times & \times & \times & & & & & & & & & \\
\times & \times & \times & \times & & & & & & & & \\
\times & \times & \times & \times & \times & & & & & & & \\
 & \times & \times & \times & \times & \times & & & & & & \\
 & & \times & \times & \times & \times & \times & & & & & \\
 & & & & & & \cdots & & & & & \\
 & & & & & & & \times & \times & \times & \times & \times \\
 & & & & & & & & \times & \times & \times & \times \\
 & & & & & & & & & \times & \times & \times
\end{pmatrix}
$$

LU-factorization

$$A = LU,$$

where

$$L = \begin{pmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ & \dots & & \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{pmatrix} \quad U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ & \dots & & \\ 0 & 0 & \dots & u_{nn} \end{pmatrix}.$$

$l_{ii} = 1$ Doolittle's method

$u_{ii} = 1$ Crout's method

if A is symmetric, i.e. $a_{ij} = a_{ji}$, then is is possible to take

$$U = L^T \qquad \Leftrightarrow \qquad A = LL^T$$

This is Cholesky's factorization

Indirect methods.

$$A\vec{x} = \vec{b}$$

Iteration: choose initial guess

$$\vec{x}^0 = (x_1^0, \ldots, x_n^0)^T.$$

Compute successively

$$\vec{x}^1 = (x_1^1, \ldots, x_n^1)^T$$

$$\vec{x}^2 = (x_1^2, \ldots, x_n^2)^T$$

and so on

Stopping criterion:

$$\|\vec{x}^k - \vec{x}^{k-1}\| < \varepsilon$$

$$\begin{array}{lllllll}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + \ldots + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + \ldots + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + \ldots + & a_{2n}x_n & = & b_3 \\
 & & \ldots & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + \ldots + & a_{nn}x_n & = & b_n.
\end{array}$$

Solving for main diagonal:

$$\begin{array}{lllllllll}
x_1 & + & \frac{a_{12}}{a_{11}}x_2 & + & \frac{a_{13}}{a_{11}}x_3 & + \ldots + & \frac{a_{1,n-1}}{a_{11}}x_{m-1} & + & \frac{a_{1n}}{a_{11}}x_n & = & \frac{b_1}{a_{11}} \\
\frac{a_{21}}{a_{22}}x_1 & + & x_2 & + & \frac{a_{23}}{a_{22}}x_3 & + \ldots + & \frac{a_{2,n-1}}{a_{22}}x_{n-1} & + & \frac{a_{2n}}{a_{22}}x_n & = & \frac{b_2}{a_{22}} \\
\frac{a_{31}}{a_{33}}x_1 & + & \frac{a_{32}}{a_{33}}x_2 & + & x_3 & + \ldots + & \frac{a_{3,n-1}}{a_{33}}x_{n-1} & + & \frac{a_{3n}}{a_{33}}x_n & = & \frac{b_3}{a_{33}} \\
 & & & & \ldots & & & & & & \\
\frac{a_{n1}}{a_{nn}}x_1 & + & \frac{a_{n2}}{a_{nn}}x_2 & + & \frac{a_{n3}}{a_{nn}}x_3 & + \ldots + & \frac{a_{n,n-1}}{a_{nn}}x_{n-1} & + & x_n & = & \frac{b_n}{a_{nn}}.
\end{array}$$

$$\begin{array}{ll}
x_1 & = \quad - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - \ldots - \frac{a_{1,n-1}}{a_{11}}x_{n-1} - \frac{a_{1n}}{a_{11}}x_n + \frac{b_1}{a_{11}} \\
x_2 & = - \frac{a_{21}}{a_{22}}x_1 - \frac{a_{23}}{a_{22}}x_3 - \ldots - \frac{a_{2,n-1}}{a_{22}}x_{n-1} - \frac{a_{2n}}{a_{22}}x_n + \frac{b_2}{a_{22}} \\
x_3 & = - \frac{a_{31}}{a_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2 - \ldots - \frac{a_{3,n-1}}{a_{33}}x_{n-1} - \frac{a_{3n}}{a_{33}}x_n + \frac{b_3}{a_{33}} \\
 & \quad \ldots \\
x_n & = - \frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}}{a_{nn}}x_2 - \frac{a_{n3}}{a_{nn}}x_3 - \ldots - \frac{a_{n,n-1}}{a_{nn}}x_{n-1} + \frac{b_n}{a_{nn}}.
\end{array}$$

$$
\begin{aligned}
x_1 &= \qquad\qquad - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - \ \ldots - \frac{a_{1,n-1}}{a_{11}}x_{n-1} - \frac{a_{1n}}{a_{11}}x_n + \frac{b_1}{a_{11}} \\
x_2 &= -\frac{a_{21}}{a_{22}}x_1 \qquad\qquad - \frac{a_{23}}{a_{22}}x_3 - \ \ldots - \frac{a_{2,n-1}}{a_{22}}x_{n-1} - \frac{a_{2n}}{a_{22}}x_n + \frac{b_2}{a_{22}} \\
x_3 &= -\frac{a_{31}}{a_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2 - \ \ldots \qquad\qquad - \frac{a_{3,n-1}}{a_{33}}x_{n-1} - \frac{a_{3n}}{a_{33}}x_n + \frac{b_3}{a_{33}} \\
&\qquad\qquad \ldots \\
x_n &= -\frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}}{a_{nn}}x_2 - \frac{a_{n3}}{a_{nn}}x_3 - \ \ldots - \frac{a_{n,n-1}}{a_{nn}}x_{n-1} \qquad\qquad + \frac{b_n}{a_{nn}}.
\end{aligned}
$$

Jacobi iteration:

$$
\begin{aligned}
x_1^k &= \qquad\qquad - \frac{a_{12}}{a_{11}}x_2^{k-1} - \frac{a_{13}}{a_{11}}x_3^{k-1} - \ \ldots - \frac{a_{1,n-1}}{a_{11}}x_{n-1}^{k-1} - \frac{a_{1n}}{a_{11}}x_n^{k-1} + \frac{b_1}{a_{11}} \\
x_2^k &= -\frac{a_{21}}{a_{22}}x_1^{k-1} \qquad\qquad - \frac{a_{23}}{a_{22}}x_3^{k-1} - \ \ldots - \frac{a_{2,n-1}}{a_{22}}x_{n-1}^{k-1} - \frac{a_{2n}}{a_{22}}x_n^{k-1} + \frac{b_2}{a_{22}} \\
x_3^k &= -\frac{a_{31}}{a_{33}}x_1^{k-1} - \frac{a_{32}}{a_{33}}x_2^{k-1} - \ \ldots \qquad\qquad - \frac{a_{3,n-1}}{a_{33}}x_{n-1}^{k-1} - \frac{a_{3n}}{a_{33}}x_n^{k-1} + \frac{b_3}{a_{33}} \\
&\qquad\qquad \ldots \\
x_n^k &= -\frac{a_{n1}}{a_{nn}}x_1^{k-1} - \frac{a_{n2}}{a_{nn}}x_2^{k-1} - \frac{a_{n3}}{a_{nn}}x_3^{k-1} - \ \ldots - \frac{a_{n,n-1}}{a_{nn}}x_{n-1}^{k-1} \qquad\qquad + \frac{b_n}{a_{nn}}.
\end{aligned}
$$

Gauss-Seidel iteration:

$$
\begin{aligned}
x_1^k &= \qquad\qquad - \frac{a_{12}}{a_{11}}x_2^{k-1} - \frac{a_{13}}{a_{11}}x_3^{k-1} - \ \ldots - \frac{a_{1,n-1}}{a_{11}}x_{n-1}^{k-1} - \frac{a_{1n}}{a_{11}}x_n^{k-1} + \frac{b_1}{a_{11}} \\
x_2^k &= -\frac{a_{21}}{a_{22}}x_1^{k} \qquad\qquad - \frac{a_{23}}{a_{22}}x_3^{k-1} - \ \ldots - \frac{a_{2,n-1}}{a_{22}}x_{n-1}^{k-1} - \frac{a_{2n}}{a_{22}}x_n^{k-1} + \frac{b_2}{a_{22}} \\
x_3^k &= -\frac{a_{31}}{a_{33}}x_1^{k} - \frac{a_{32}}{a_{33}}x_2^{k} - \ \ldots \qquad\qquad - \frac{a_{3,n-1}}{a_{33}}x_{n-1}^{k-1} - \frac{a_{3n}}{a_{33}}x_n^{k-1} + \frac{b_3}{a_{33}} \\
&\qquad\qquad \ldots \\
x_n^k &= -\frac{a_{n1}}{a_{nn}}x_1^{k} - \frac{a_{n2}}{a_{nn}}x_2^{k} - \frac{a_{n3}}{a_{nn}}x_3^{k} - \ \ldots - \frac{a_{n,n-1}}{a_{nn}}x_{n-1}^{k} \qquad\qquad + \frac{b_n}{a_{nn}}.
\end{aligned}
$$

Convergence

$\|\vec{x}^k - \vec{x}^*\| \to 0$ as $k \to \infty$ where $\vec{x}^*$–exact solution

Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

is strictly diagonally dominant, if for any $i = 1, \dots, n$ the inequality

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|.$$

is valid.

If A is diagonally dominant, then Jacobi and Gauss-Seidel iterations converge in case of any initial guess.

Relaxation method.

Formula for i-th row in Gauss-Seidel iteration:

$$x_i^k = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n} a_{ij}x_j^{k-1}\right]$$

Stepsize of going from x_i^{k-1} to x_i^k:

$$x_i^k - x_i^{k-1} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i}^{n} a_{ij}x_j^{k-1}\right]$$

Change this stepsize by factor $\omega > 0$. Here ω is so-called relaxation parameter. We obtain

$$x_i^k - x_i^{k-1} = \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i}^{n} a_{ij}x_j^{k-1}\right]$$

Express again x_i^k:

$$x_i^k = (1-\omega)x_i^{k-1} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n} a_{ij}x_j^{k-1}\right].$$

$0 < \omega < 1$ - underrelaxation

$\omega > 1$ - overrelaxation

Formulas of numerical differentiation.

Taylor's formula of $k+1$-times continuously differentiable function u at x_i:

$$u(x) = \underbrace{u(x_i) + u'(x_i)(x - x_i) + \frac{u''(x_i)}{2!}(x - x_i)^2 + \ldots + \frac{u^{(k)}(x_i)}{k!}(x - x_i)^k}_{\text{Taylor's polynomial}} +$$

$$+ \underbrace{\frac{u^{(k+1)}(\xi)}{(k+1)!}(x - x_i)^{k+1}}_{\text{remainder}}.$$

Forward difference formula:

$$u'(x_i) \;=\; \frac{u(x_{i+1}) - u(x_i)}{h} - \frac{u''(\xi)}{2}h\,, \quad \xi \in (x_i, x_{i+1}) \tag{1}$$

$$u'(x_i) \;\approx\; \frac{u(x_{i+1}) - u(x_i)}{h} \tag{2}$$

Backward difference formula:

$$u'(x_i) \;=\; \frac{u(x_i) - u(x_{i-1})}{h} + \frac{u''(\xi)}{2}h\,, \quad \xi \in (x_{i-1}, x_i) \tag{3}$$

$$u'(x_i) \;\approx\; \frac{u(x_i) - u(x_{i-1})}{h} \tag{4}$$

Symmetric difference formula:

$$u'(x_i) = \frac{u(x_{i+1}) - u(x_{i-1})}{2h} - \frac{u'''(\xi) + u'''(\eta)}{12} h^2, \quad (5)$$

$$\xi \in (x_i, x_{i+1}),\ \eta \in (x_{i-1}, x_i) \quad (6)$$

$$u'(x_i) \approx \frac{u(x_{i+1}) - u(x_{i-1})}{2h} \quad (7)$$

Difference formula for 2nd order derivative:

$$u''(x_i) = \frac{u(x_{i-1}) + u(x_{i+1}) - 2u(x_i)}{h^2} - \frac{u^{iv}(\xi) + u^{iv}(\eta)}{24}h^2, \quad (8)$$

$$\xi \in (x_i, x_{i+1}),\ \eta \in (x_{i-1}, x_i)$$

$$u''(x_i) \approx \frac{u(x_{i-1}) + u(x_{i+1}) - 2u(x_i)}{h^2} \quad (9)$$

Formulas of numerical integration.

Definition of the integral $\int_0^L f(x)dx$.

Gridpoints: $x_i = ih$, $i = 0, \ldots, n$, $h = L/n$.
Additional points: $p_i \in [x_{i-1}, x_i]$.

$\int_0^L f(x)dx = \lim\limits_{h\to 0} h \sum\limits_{i=1}^{n} f(p_i)$.

This means that

$$\int_0^L f(x)dx \approx h \sum_{i=1}^{n} f(p_i).$$

Left rectangular rule:

$$\int_0^L f(x)dx \approx S_n = h \sum_{i=1}^{n} f(x_{i-1}).$$

Central rectangular rule:

$$\int_0^L f(x)dx \approx S_n = h \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right).$$

Right rectangular rule:

$$\int_0^L f(x)dx \approx S_n = h \sum_{i=1}^{n} f(x_i).$$

For all rectangular rules,

$$\left| S_n - \int_0^L f(x)dx \right| \leq Ch, \qquad C \text{ - constant.}$$

Trapezoidal rule:

$$\int_0^L f(x)dx \approx S_n =$$
$$\frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

For trapezoidal rule,

$$\left| S_n - \int_0^L f(x)dx \right| \leq Ch^2, \qquad C \text{ - constant.}$$

FDM for 1D problem

$$
\begin{aligned}
&-pu''(x) + qu(x) = f(x), \quad x \in (0, L),\\
&u(0) = a, \quad u'(L) = b.
\end{aligned}
$$

$x_i = ih$, $i = 0, \ldots, n$, $\qquad h = L/n$.

$$-\frac{p}{h^2}u_{i-1} + \left(\frac{2p}{h^2} + q\right)u_i - \frac{p}{h^2}u_{i+1} = f(x_i),\ i = 1, \ldots, n-1, \tag{10}$$

$$u_0 = a, \tag{11}$$

$$-\frac{u_{n-1}}{2h} + \frac{u_{n+1}}{2h} = b, \tag{12}$$

$$-\frac{p}{h^2}u_{n-1} + \left(\frac{2p}{h^2} + q\right)u_n - \frac{p}{h^2}u_{n+1} = f(x_n). \tag{13}$$

Here $u_i \approx u(x_i)$, $i = 0, \ldots, n+1$.

Additional gridpoint: $x_{n+1} = L + h$.

Equivalent formulation of the equation

The equation

$$-(p(x)u'(x))' + q(x)u(x) = f(x)$$

holds in the interval $(0, L)$ if and only if

$$\int_0^L (-(pu')' + qu - f)v\,dx = 0 \qquad \forall v \in L_2(0, L).$$

Weighted residual method for 1D problem

Approximate solution:

$$u_h(x) = \sum_{i=1}^{n} u_i \varphi_i(x) \tag{14}$$

$\varphi_1, \ldots, \varphi_n$ - basis functions

System of equations:

$$\int_0^L \Big[-(p(x)u_h'(x))' + q(x)u_h(x) - f(x)\Big] v_i(x)dx = 0, \tag{15}$$

$$i = 1, \ldots, n.$$

$v_1, \ldots, v_n$ - test functions

Variational formulations of 1D problem

$$\int_0^L (-(pu')' + qu - f)v dx = 0 \qquad \forall v \in L_2(0, L).$$

Assume that the test function v is differentiable.

$$-pu'(L)v(L)+pu'(0)v(0)+\int_0^L pu'v'dx+\int_0^L (qu-f)vdx = 0.$$

Introduce the boundary conditions

$$u(0) = a, \quad u'(L) = b.$$

The boundary value $u'(0)$ is not given! Additional restriction to the test function:

$$v(0) = 0.$$

Then

$$-pbv(L) + \int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0.$$

Variational formulation of the problem with boundary conditions $u(0) = a, \quad u'(L) = b$:
Find a function u that satisfies the boundary condition $u(0) = a$ and the equation

$$-pbv(L) + \int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0 \quad (16)$$

for any differentiable test function v such that $v(0) = 0$.

Additional explanation of this variational formulation.

If p is continuously differentiable, q and f are continuous and $p > 0$ then the original problem

$$(-(p(x)u(x)')' + q(x)u(x) - f(x) = 0,\ x \in (0, L), \quad u(0) = a,\ u'(L) = b \qquad (17)$$

has a unique twice continuously differentiable solution u. This solution is called the classical solution.

However, in some important cases the classical solution does not exist, for example when p or q have jumps. Therefore, it makes sense to generalize the concept of the solution. For this purpose we introduce the variational problem. Variational problem enables to consider solutions in the wider space

$$H^1(0, L) = \{u : u - \text{continuous},\ u' \in L^2(0, L)\}.$$

Before going to the variational problem, let us take a look on properties of functions $u \in H^1(0, L)$. Derivatives of functions that belong to this space are integrable but may have discontinuities, for instance jumps. This means that $u'(t)$ may not have a meaning in every particular point t of the interval $[0, L]$. Moreover, functions of the class $H^1(0, L)$ may not have second order derivatives.

An example of a function that belongs to $H^1(0, L)$ is the linear continuous spline u_h defined on a grid $x_i = ih$, $i = 0, \ldots, n$ $(h = L/n)$ that we will use in FEM below. The derivative u_h' is a piecewise constant function that has jumps at gridpoints x_i, hence it has no meaning at gridpoints (more precisely: we have infinitely many options to define the value of u_h' at $x = x_i$).

Let us return to the procedure of derivation of the variational problem and consider it in more details. After multiplying the differential equation by an arbitrary test function $v \in L_2(0, L)$ and integrating we have

$$\int_0^L (-(pu')' + qu - f)v dx = 0.$$

Now we restrict the class of test functions to $v \in H^1(0, L)$ and after integration by parts obtain

$$-pu'(L)v(L) + pu'(0)v(0) + \int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0.$$

Next we use the second boundary condition to replace $u'(L)$ by b:

$$-pbv(L) + pu'(0)v(0) + \int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0.$$

This relation is almost acceptable for functions u in the class $H^1(0, L)$. The involved integrals exist for $u \in H^1(0, L)$. Only the term $pu'(0)v(0)$ is not OK, because the value $u'(0)$ may not exist for $u \in H^1(0, L)$. Therefore, we make the further restriction of the space of test functions assuming that

$$v \in H_o^1(0, L) = \{v \in H^1(0, L) : v(0) = 0\}.$$

Then we obtain

$$-pbv(L) + \int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0.$$

This relation is defined for all $u \in H^1(0, L)$.

Summing up, we formulate the following (variational) problem: Find a function $u \in H^1(0, L)$ that satisfies the boundary condition $u(0) = a$ and the equation

$$-pbv(L) + \int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0 \tag{18}$$

for any test function $v \in H_o^1(0, L)$.

It is clear that any classical solution of the original boundary value problem is a solution of the variational problem. This directly follows from the presented computations. But we have to show that such an assertion holds also vice versa, i.e. a twice differentiable function u that solves the variational problem, solves the original problem. This is important because we have made essential restrictions on the test functions and such restrictions may have increased the number of solutions.

So, let a twice continuously differentiable function u solve the variational problem. Then we invert the integration by parts and get

$$\int_0^L (-(pu')' + qu - f)vdx = 0.$$

This holds for any $v \in H_o^1(0, L)$. Since $H_o^1(0, L)$ is a dense subspace of $L_2(0, L)$, the latter equality is valid for all $v \in L_2(0, L)$, too. This implies $-(p(x)u'(x))' + q(x)u(x) - f(x) = 0$, $x \in (0, L)$. Hence, the differential equation is satisfied. The boundary condition $u(0) = a$ is automatically satisfied because it is assumed in the variational problem. It remains to show that the boundary condition $u'(L) = b$ is also valid.

To show that $u'(L) = b$, we choose the following sequence of test functions v_n in the space $H_o^1(0, L)$:

$$v_n(x) = e^{-n\frac{L-x}{x}}.$$

The functions v_n have the following properties: $v_n(L) = 1$ and

$$\int_0^L v_n'(x)z(x)dx \to z(L), \int_0^L v_n(x)z(x)dx \to 0 \quad \text{as} \quad n \to \infty$$

for continuously differentiable z. Plugging v_n into (18) and passing to the limit $n \to \infty$ we obtain the equality $-pb + pu'(L) = 0$. This implies $u'(L) = b$.

Finally, we mention that the further restriction $v(L) = 0$ on the test functions increases the number of solutions. A twice differentiable solution u of the variational problem may not satisfy the boundary condition $u'(L) = b$.

Variational formulation of the problem with boundary conditions $u(0) = a, \quad u(L) = b:$
Find a function u that satisfies the boundary conditions $u(0) = a$, $u(L) = b$ and the equation

$$\int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0 \tag{19}$$

for any test function v such that $v(0) = v(L) = 0$.

Variational formulation of the problem with boundary conditions $u'(0) = a, \quad u'(L) = b:$
Find a function u that satisfies the equation

$$\begin{aligned} - pbv(L) + pav(0) + \int_0^L pu'v'dx \\ + \int_0^L (qu - f)vdx = 0 \end{aligned} \tag{20}$$

for any test function v.

Galerkin FEM for 1D problems

Firstly, we follow the variational formulation of the problem with boundary conditions $u(0) = a, \quad u'(L) = b$:
Find a function u that satisfies the boundary condition $u(0) = a$ and the equation

$$-pbv(L) + \int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0.$$

for any test function v such that $v(0) = 0$.

Shape functions $\varphi_1, \ldots, \varphi_n$.

$$\varphi_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere.} \end{cases} \tag{21}$$

Approximate solution is searched in the form

$$u_h(x) = \sum_{j=0}^{n} u_j \varphi_j(x) = a\varphi_0(x) + \sum_{j=1}^{n} u_j \varphi_j(x). \tag{22}$$

The numbers $u_1, \ldots, u_n$ are to be determined.

Test functions: $\varphi_1, \ldots, \varphi_n$.

$$-pb\varphi_i(L) + \int_0^L pu_h'\varphi_i'dx + \int_0^L (qu_h - f)\varphi_i dx = 0, \quad i = 1, \ldots, n.$$

$$
\begin{aligned}
-pb\varphi_i(L) &+ \int_0^L p\left[a\varphi_0' + \sum_{j=1}^n u_j\varphi_j'\right]\varphi_i' dx \\
&+ \int_0^L \left(q\left[a\varphi_0 + \sum_{j=1}^n u_j\varphi_j\right] - f\right)\varphi_i dx = 0, \quad i = 1, \dots, n.
\end{aligned}
$$

This leads to the linear system of equations

$$
\begin{aligned}
\sum_{j=1}^n u_j &\left[\int_0^L p\varphi_j'\varphi_i' dx + \int_0^L q\varphi_j\varphi_i dx\right] \\
&= \int_0^L f\varphi_i dx + pb\varphi_i(L) - a\left[\int_0^L p\varphi_0'\varphi_i' dx + \int_0^L q\varphi_0\varphi_i dx\right], \\
&i = 1, \dots, n.
\end{aligned} \tag{23}
$$

Analogously we obtain systems of equations in case of the boundary conditions $u(0) = a$, $u(L) = b$.

Recall the variational formulation of this problem:

Find a function u that satisfies the boundary conditions $u(0) = a$, $u(L) = b$ and the equation

$$\int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0$$

for any test function v such that $v(0) = v(L) = 0$. Approximate solution is searched in the form

$$u_h(x) = a\varphi_0(x) + \sum_{j=1}^{n-1} u_j\varphi_j(x) + b\varphi_n(x).$$

Test functions: $\varphi_1, \ldots, \varphi_{n-1}$. We obtain the system

$$\sum_{j=1}^{n-1} u_j \left[\int_0^L p\varphi_j'\varphi_i'dx + \int_0^L q\varphi_j\varphi_i dx\right] \tag{24}$$
$$= \int_0^L f\varphi_i dx - a\left[\int_0^L p\varphi_0'\varphi_i'dx + \int_0^L q\varphi_0\varphi_i dx\right]$$
$$-b\left[\int_0^L p\varphi_n'\varphi_i'dx + \int_0^L q\varphi_n\varphi_i dx\right], \quad i = 1, \ldots, n-1.$$

Let us deduce the system for the boundary conditions $u'(0) = a$, $u'(L) = b$, too.

Variational formulation of this problem:

Find a function u that satisfies the equation

$$
\begin{aligned}
-\, pbv(L) + pav(0) + \int_0^L pu'v'dx \\
+ \int_0^L (qu - f)vdx = 0
\end{aligned}
$$

for any test function v. Approximate solution is searched in the form

$$
u_h(x) = \sum_{j=0}^{n} u_j \varphi_j(x).
$$

Test functions: $\varphi_0, \ldots, \varphi_n$. We obtain the system

$$
\begin{aligned}
\sum_{j=0}^{n} u_j \left[\int_0^L p\varphi_j'\varphi_i' dx + \int_0^L q\varphi_j\varphi_i dx \right] \\
= \int_0^L f\varphi_i dx + pb\varphi_i(L) - pa\varphi_i(0), \\
i = 0, \ldots, n.
\end{aligned}
\tag{25}
$$

Auxiliary formulas in case $h = x_i - x_{i-1}$ - constant.

$$\varphi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{h} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere.} \end{cases}$$

$$\varphi_i'(x) = \begin{cases} \frac{1}{h} & \text{for } x \in [x_{i-1}, x_i] \\ -\frac{1}{h} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere.} \end{cases}$$

Therefore,

$$\int_0^L \varphi_j' \varphi_i' dx = \begin{cases} \frac{2}{h} & \text{for } j = i \notin \{0; n\} \\ \frac{1}{h} & \text{for } j = i \in \{0; n\} \\ -\frac{1}{h} & \text{for } j = i-1 \text{ and } j = i+1 \\ 0 & \text{elsewhere.} \end{cases}$$

Application of trapezoidal rule:

$$\int_{x_{i-1}}^{x_{i+1}} F(x)dx \approx \frac{h}{2}\left[F(x_{i-1}) + 2F(x_i) + F(x_{i+1})\right] \quad \text{3-point formula}$$

$$\int_{x_{i-1}}^{x_i} F(x)dx \approx \frac{h}{2}\left[F(x_{i-1}) + F(x_i)\right] \quad \text{2-point formula}$$

Therefore

$$\int_0^L q\varphi_j\varphi_i dx \approx \begin{cases} hq(x_i) & \text{for } j = i \notin \{0; n\} \\ \frac{h}{2}q(x_i) & \text{for } j = i \in \{0; n\} \\ 0 & \text{elsewhere.} \end{cases}$$

$$\int_0^L f\varphi_i dx \approx \begin{cases} hf(x_i) & \text{for } i \notin \{0; n\} \\ \frac{h}{2}f(x_i) & \text{for } i \in \{0; n\} \\ 0 & \text{elsewhere.} \end{cases}$$

Example 3.

$$\begin{cases} -(pu')'(x) = 10x(2-x), \quad x \in (0,2), \\ u(0) = 1, \quad u'(2) = 1. \end{cases}$$

$$p(x) = \begin{cases} 1 & \text{for } x \in (0,1) \\ 2 & \text{for } x \in (1,2). \end{cases}$$

Number of meshpoints: $n = 200$

System of equations:

$$\sum_{j=1}^{n} u_j \int_0^2 p\varphi_j'\varphi_i' dx$$
$$= 10 \int_0^2 x(2-x)\varphi_i dx + 2\varphi_i(2) - \int_0^2 p\varphi_0'\varphi_i' dx, \quad i = 1, \ldots, n.$$

It has the form

$$A\vec{u} = \vec{y},$$

where

$$A_{11} = \frac{2}{h} \quad A_{12} = -\frac{1}{h}$$

$$A_{i,i-1} = \begin{cases} -\frac{1}{h} & \text{for } i \leq 100 \\ -\frac{2}{h} & \text{for } i > 100 \end{cases} \qquad A_{i,i} = \begin{cases} \frac{2}{h} & \text{for } i < 100 \\ \frac{3}{h} & \text{for } i = 100 \\ \frac{4}{h} & \text{for } i > 100 \end{cases}$$

$$A_{i,i+1} = \begin{cases} -\frac{1}{h} & \text{for } i < 100 \\ -\frac{2}{h} & \text{for } i \geq 100 \end{cases} \qquad \text{for } i = 2, \ldots, n-1$$

$$A_{n,n-1} = -\frac{2}{h} \quad A_{nn} = \frac{2}{h}$$

$$y_1 = \frac{1}{h}$$

$$y_i = 10hx_i(2 - x_i) \qquad \text{for } i = 2, \ldots, n-1$$

$$y_n = 2$$

FVM for 1D problem

$$\int_0^L \left(-(pu_h')' + qu_h - f\right) 1_{\omega_i} dx = 0, \quad i = 1, \ldots, n, \quad (26)$$

$$1_{\omega_i}(x) = \begin{cases} 1 & \text{for } x \in \omega_i, \\ 0 & \text{for } x \notin \omega_i \end{cases}$$

$$\omega_0 = (x_0, x_{\frac{1}{2}}), \quad \omega_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}),\ i = 1, \ldots, n-1, \quad \omega_n = (x_{n-\frac{1}{2}}, x_n).$$

$$x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$$

$$\int_{x_0}^{x_{\frac{1}{2}}} (-(pu_h')' + qu_h - f)dx = 0,$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (-(pu_h')' + qu_h - f)dx = 0, \quad i = 1, \ldots, n-1,$$

$$\int_{x_{n-\frac{1}{2}}}^{x_n} (-(pu_h')' + qu_h - f)dx = 0.$$

Basic equations of FVM:

$$-(pu_h)'(x_{\frac{1}{2}}) + (pu_h')(x_0) + \int_{x_0}^{x_{\frac{1}{2}}} (qu_h - f)dx = 0, \quad (27)$$

$$-(pu_h')(x_{i+\frac{1}{2}}) + (pu_h')(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0, \quad (28)$$

$$i = 1, \ldots, n-1,$$

$$-(pu_h')(x_n) + (pu_h')(x_{n-\frac{1}{2}}) + \int_{x_{n-\frac{1}{2}}}^{x_n} (qu_h - f)dx = 0. \quad (29)$$

$$u_h = \sum_{j=0}^{n} u_j \varphi_j \quad (30)$$

φ_j - the same shape functions as in FEM.

A system of equations of FVM depends on boundary conditions.

Boundary conditions: $u(0) = a$, $u'(L) = b$.
Solution is searched in the form

$$u_h = a\varphi_0 + \sum_{j=1}^{n} u_j \varphi_j \tag{31}$$

and the system of equations is

$$-(pu_h')(x_{i+\frac{1}{2}}) + (pu_h')(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0, \tag{32}$$

$$i = 1, \ldots, n-1,$$

$$-p(x_n)b + (pu_h')(x_{n-\frac{1}{2}}) + \int_{x_{n-\frac{1}{2}}}^{x_n} (qu_h - f)dx = 0. \tag{33}$$

Boundary conditions: $u(0) = a$, $u(L) = b$.
Solution is searched in the form

$$u_h = a\varphi_0 + \sum_{j=1}^{n-1} u_j \varphi_j + b\varphi_n \tag{34}$$

and the system of equations is

$$-(pu_h')(x_{i+\frac{1}{2}}) + (pu_h')(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0, \tag{35}$$

$$i = 1, \ldots, n-1.$$

Boundary conditions: $u'(0) = a$, $u'(L) = b$.
Solution is searched in the form

$$u_h = \sum_{j=0}^{n} u_j \varphi_j \tag{36}$$

and the system of equations is

$$-(pu_h)'(x_{\frac{1}{2}}) + p(x_0)a + \int_{x_0}^{x_{\frac{1}{2}}} (qu_h - f)dx = 0, \tag{37}$$

$$-(pu_h')(x_{i+\frac{1}{2}}) + (pu_h')(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0, \tag{38}$$

$$i = 1, \dots, n-1,$$

$$-p(x_n)b + (pu_h')(x_{n-\frac{1}{2}}) + \int_{x_{n-\frac{1}{2}}}^{x_n} (qu_h - f)dx = 0. \tag{39}$$

Auxiliary formulas:

$$u_h'(x_{i-\frac{1}{2}}) = \frac{1}{h}(u_i - u_{i-1}), \quad u_h'(x_{i+\frac{1}{2}}) = \frac{1}{h}(u_{i+1} - u_i) \tag{40}$$

By means of the rectangular rule

$$\int_{x_0}^{x_{\frac{1}{2}}} w(x)dx \approx (x_{\frac{1}{2}} - x_0)w(x_0) = \frac{h}{2}w(x_0), \tag{41}$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x)dx \approx (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}})w(x_i) = hw(x_i), \tag{42}$$

$$\int_{x_{n-\frac{1}{2}}}^{x_n} w(x)dx \approx (x_n - x_{n-\frac{1}{2}})w(x_n) = \frac{h}{2}w(x_n). \tag{43}$$

Let's return to the systems of equations.

A. Boundary conditions: $u(0) = a$, $u'(L) = b$.

Solution is searched in the form

$$u_h = a\varphi_0 + \sum_{j=1}^{n} u_j \varphi_j \tag{44}$$

and the system of equations is

$$-(pu_h')(x_{i+\frac{1}{2}}) + (pu_h')(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0,$$

$$i = 1, \ldots, n-1,$$

$$-p(x_n)b + (pu_h')(x_{n-\frac{1}{2}}) + \int_{x_{n-\frac{1}{2}}}^{x_n} (qu_h - f)dx = 0.$$

Using the formula of u_h and auxiliary formulas in the previous slide we have

$$-p(x_{\frac{3}{2}})\frac{1}{h}(u_2 - u_1) + p(x_{\frac{1}{2}})\frac{1}{h}(u_1 - a) + h(qu_h - f)(x_1) = 0,$$

$$-p(x_{i+\frac{1}{2}})\frac{1}{h}(u_{i+1} - u_i) + p(x_{i-\frac{1}{2}})\frac{1}{h}(u_i - u_{i-1}) + h(qu_h - f)(x_i) = 0,$$

$$i = 2, \ldots, n-1,$$

$$-p(x_n)b + p(x_{n-\frac{1}{2}})\frac{1}{h}(u_n - u_{n-1}) + \frac{h}{2}(qu_h - f)(x_n) = 0.$$

Since $u_h(x_i) = u_i$, this results in the system

$$\left[\frac{p(x_{\frac{1}{2}}) + p(x_{\frac{3}{2}})}{h} + hq(x_1)\right] u_1 - \frac{p(x_{\frac{3}{2}})}{h} u_2 = hf(x_1) + \frac{p(x_{\frac{1}{2}})}{h} a, \quad (45)$$

$$-\frac{p(x_{i-\frac{1}{2}})}{h} u_{i-1} + \left[\frac{p(x_{i-\frac{1}{2}}) + p(x_{i+\frac{1}{2}})}{h} + hq(x_i)\right] u_i - \frac{p(x_{i+\frac{1}{2}})}{h} u_{i+1} =$$
$$= hf(x_i), \quad i = 2, \ldots, n-1, \quad (46)$$

$$-\frac{p(x_{n-\frac{1}{2}})}{h} u_{n-1} + \left[\frac{p(x_{n-\frac{1}{2}})}{h} + \frac{h}{2} q(x_n)\right] u_n = \frac{h}{2} f(x_n) + p(x_n) b \quad (47)$$

B. Boundary conditions: $u(0) = a$, $u(L) = b$.
Solution is searched in the form

$$u_h = a\varphi_0 + \sum_{j=1}^{n-1} u_j\varphi_j + b\varphi_n \tag{48}$$

and the system of equations is

$$-(pu_h')(x_{i+\frac{1}{2}}) + (pu_h')(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0,$$

$$i = 1, \ldots, n-1.$$

The system for $u_1, \ldots, u_{n-1}$:

$$\left[\frac{p(x_{\frac{1}{2}}) + p(x_{\frac{3}{2}})}{h} + hq(x_1)\right] u_1 - \frac{p(x_{\frac{3}{2}})}{h} u_2 = hf(x_1) + \frac{p(x_{\frac{1}{2}})}{h} a, \quad (49)$$

$$-\frac{p(x_{i-\frac{1}{2}})}{h} u_{i-1} + \left[\frac{p(x_{i-\frac{1}{2}}) + p(x_{i+\frac{1}{2}})}{h} + hq(x_i)\right] u_i - \frac{p(x_{i+\frac{1}{2}})}{h} u_{i+1} =$$

$$= hf(x_i), \quad i = 2, \ldots, n-2, \quad (50)$$

$$-\frac{p(x_{n-\frac{3}{2}})}{h} u_{n-2} + \left[\frac{p(x_{n-\frac{3}{2}}) + p(x_{n-\frac{1}{2}})}{h} + hq(x_{n-1})\right] u_{n-1} =$$

$$= hf(x_{n-1}) + \frac{p(x_{n-\frac{1}{2}})}{h} b \quad (51)$$

C. Boundary conditions: $u'(0) = a$, $u'(L) = b$.
Solution is searched in the form

$$u_h = \sum_{j=0}^{n} u_j \varphi_j \tag{52}$$

and the system of equations is

$$-(pu_h)'(x_{\frac{1}{2}}) + p(x_0)a + \int_{x_0}^{x_{\frac{1}{2}}} (qu_h - f)dx = 0,$$

$$-(pu'_h)(x_{i+\frac{1}{2}}) + (pu'_h)(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0,$$

$$i = 1, \ldots, n-1,$$

$$-p(x_n)b + (pu'_h)(x_{n-\frac{1}{2}}) + \int_{x_{n-\frac{1}{2}}}^{x_n} (qu_h - f)dx = 0.$$

The system for $u_0, \ldots, u_n$:

$$\left[\frac{p(x_{\frac{1}{2}})}{h} + \frac{h}{2} q(x_0)\right] u_0 - \frac{p(x_{\frac{1}{2}})}{h} u_1 = \frac{h}{2} f(x_0) - p(x_0) a, \tag{53}$$

$$-\frac{p(x_{i-\frac{1}{2}})}{h} u_{i-1} + \left[\frac{p(x_{i-\frac{1}{2}}) + p(x_{i+\frac{1}{2}})}{h} + h q(x_i)\right] u_i - \frac{p(x_{i+\frac{1}{2}})}{h} u_{i+1} =$$
$$= h f(x_i), \quad i = 1, \ldots, n-1, \tag{54}$$

$$-\frac{p(x_{n-\frac{1}{2}})}{h} u_{n-1} + \left[\frac{p(x_{n-\frac{1}{2}})}{h} + \frac{h}{2} q(x_n)\right] u_n = \frac{h}{2} f(x_n) + p(x_n) b \tag{55}$$

FDM for 2D problem

$$-\Delta u + qu = f \quad \text{in} \quad (0, L_x) \times (0, L_y) \qquad (56)$$

Grid:

$x_i = ih_x$, $i = 0, \ldots, n$, $x_n = L_x$,
$y_j = jh_y$, $j = 0, \ldots, m$, $y_m = L_y$.

$u_{ij} \approx u(x_i, y_j)$

5-point difference scheme

$$\begin{aligned} &\left(\frac{2}{h_x^2} + \frac{2}{h_y^2} + q\right) u_{ij} \qquad (57) \\ &\quad - \frac{1}{h_x^2} u_{i-1,j} - \frac{1}{h_x^2} u_{i+1,j} - \frac{1}{h_y^2} u_{i,j-1} - \frac{1}{h_y^2} u_{i,j+1} = f(x_i, y_j), \\ &\quad i = 1, \ldots, n-1, \ j = 1, \ldots, m-1. \end{aligned}$$

A. First kind boundary conditions

$$u(x,0) = g_1(x), \;\; u(x, L_y) = g_2(x),$$
$$u(0,y) = g_3(y), \;\; u(L_x, y) = g_4(y).$$

Equations corresponding to these conditions:

$$\begin{aligned} &u_{i0} = g_1(x_i),\, u_{im} = g_2(x_i),\, i = 0, \ldots, n, \\ &u_{0j} = g_3(y_j),\, u_{nj} = g_4(y_j),\, j = 0, \ldots, m. \end{aligned} \tag{58}$$

B. Mixed type boundary conditions

$$u(x,0) = g_1(x), \quad u(x,L_y) = g_2(x), \qquad u(0,y) = g_3(y), \tag{59}$$

$$u_x(L_x,y) = \gamma(y). \tag{60}$$

Equations corresponding to conditions (59):

$$u_{i0} = g_1(x_i),\ u_{im} = g_2(x_i),\ i = 0,\ldots,n,$$

$$u_{0j} = g_3(y_j),\ j = 0,\ldots,m. \tag{61}$$

Equations corresponding to condition (60):

$$u_{n+1,j} - u_{n-1,j} = 2h_x\,\gamma(y_j),\ j = 1,\ldots,m-1. \tag{62}$$

Here $u_{n+1,j} \approx u(x_{n+1},y_j)$, $x_{n+1} = L_x + h$.

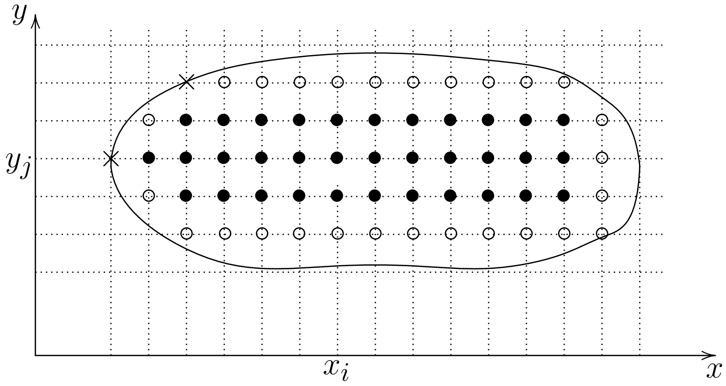
Additional main equations at the right boundary:

$$\left(\frac{2}{h_x^2} + \frac{2}{h_y^2} + q\right)u_{nj} \tag{63}$$

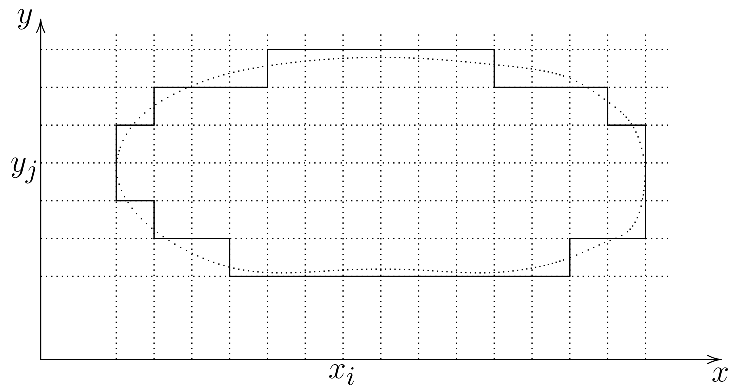
$$- \frac{1}{h_x^2}u_{n-1,j} - \frac{1}{h_x^2}u_{n+1,j} - \frac{1}{h_y^2}u_{n,j-1} - \frac{1}{h_y^2}u_{n,j+1} = f(x_n,y_j),$$

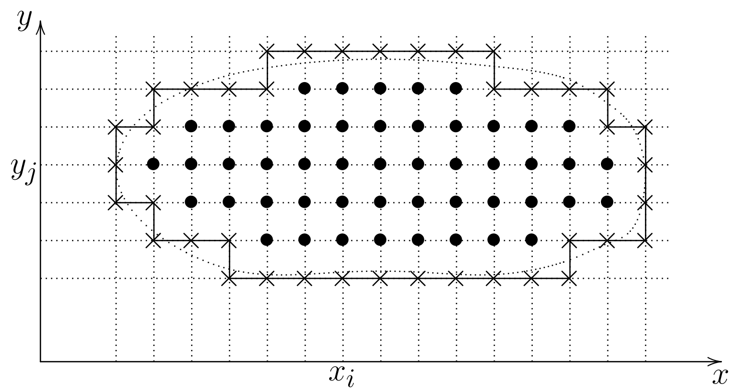
$$j = 1,\ldots,m-1.$$

Dirichlet problem in a nonrectangular domain



Approximation of a nonrectangular domain





Some methods for Cauchy problems for systems of ODE

$$\vec{u}\,'(t) = \vec{F}(t, \vec{u}(t)), \quad t \in (0, T), \quad \vec{u}(0) = \vec{u}^0. \qquad (64)$$

Here

$$\vec{u} = (u_1, \ldots, u_n), \quad \vec{F}(t, \vec{u}) = (F_1(t, \vec{u}), \ldots, F_n(t, \vec{u})),$$

$$\vec{u}^0 = (u_1^0, \ldots, u_n^0) \in \mathbb{R}^n.$$

Grid: $t_0 = 0$, $t_1 = \tau$, $t_2 = 2\tau$, ..., $t_k = k\tau$, ...,

$\tau > 0$ - stepsize

Forward Euler method:

$$\frac{\vec{u}(t_{k+1}) - \vec{u}(t_k)}{\tau} = \vec{F}(t_k, \vec{u}(t_k)) + O(\tau) \qquad (65)$$

$$\vec{u}^{k+1} = \vec{u}^k + \tau \vec{F}(t_k, \vec{u}^k), \qquad (66)$$

where $\vec{u}^k \approx \vec{u}(t_k)$.

Backward Euler method:

$$\frac{\vec{u}(t_{k+1}) - \vec{u}(t_k)}{\tau} = \vec{F}(t_{k+1}, \vec{u}(t_{k+1})) + O(\tau) \qquad (67)$$

$$\vec{u}^{k+1} - \tau \vec{F}(t_{k+1}, \vec{u}^{k+1}) = \vec{u}^k \qquad (68)$$

Method of trapezoidal rule:

$$\frac{\vec{u}(t_{k+1}) - \vec{u}(t_k)}{\tau} = \frac{1}{2}\Bigg[\vec{F}(t_k, \vec{u}(t_k)) + \vec{F}(t_{k+1}, \vec{u}(t_{k+1}))\Bigg] + O(\tau^2) \quad (69)$$

$$\vec{u}^{k+1} = \vec{u}^k + \frac{\tau}{2}\Bigg[\vec{F}(t_k, \vec{u}^k) + \tau\vec{F}(t_{k+1}, \vec{u}^{k+1})\Bigg]. \qquad (70)$$

Discretization of heat equation

$$u_t(x,t) - p u_{xx}(x,t) = f(x,t), \qquad x \in (0,L),\ t \in (0,T), \tag{71}$$

$$u(0,t) = a(t), \quad u(L,t) = b(t), \quad t \in (0,T), \tag{72}$$

$$u(x,0) = \varphi(x), \quad x \in [0,L]. \tag{73}$$

Consistency conditions: $\varphi(0) = a(0)$, $\varphi(L) = b(0)$.

Method of lines:

$$x_i = ih, \quad h = L/n.$$

$$\widehat{u}_i(t) \approx u(x_i,t) \tag{74}$$

$$\left.\begin{aligned}
&\widehat{u}_i{}'(t) = p\,\frac{\widehat{u}_{i-1}(t) + \widehat{u}_{i+1}(t) - 2\widehat{u}_i(t)}{h^2} + f(x_i,t), \quad t \in (0,T),\\
&\qquad i = 1,\ldots,n-1,\\
&\widehat{u}_0(t) = a(t),\ \ \widehat{u}_n(t) = b(t), \quad t \in (0,T),\\
&\widehat{u}_i(0) = \varphi(x_i), \quad i = 0,\ldots,n.
\end{aligned}\right\} \tag{75}$$

Local truncation error: $O(h^2)$.

Explicit scheme:

$$u_i^{k+1} = \left(1 - \frac{2p\tau}{h^2}\right) u_i^k + \frac{p\tau}{h^2}\left(u_{i-1}^k + u_{i+1}^k\right) + \tau f(x_i, t_k), \quad (76)$$

$$i = 1, \ldots, n-1$$

$$u_0^{k+1} = a(t_{k+1}), \quad u_n^{k+1} = b(t_{k+1}), \quad (77)$$

for $k = 0, 1, \ldots, l-1$ and

$$u_i^0 = \varphi(x_i), \quad i = 0, \ldots, n. \quad (78)$$

Here $u_i^k \approx \widehat{u}_i(t_k) \approx u(x_i, t_k)$.

Local truncation error: $O(h^2 + \tau)$. Conditionally stable. Stability condition:

$$\frac{2p\tau}{h^2} \leq 1.$$

Implicit scheme:

$$-\frac{p\tau}{h^2}u_{i-1}^{k+1} + \left(1 + \frac{2p\tau}{h^2}\right)u_i^{k+1} - \frac{p\tau}{h^2}u_{i+1}^{k+1} = u_i^k + \tau f(x_i, t_{k+1}), \quad (79)$$

$$i = 1, \ldots, n-1$$

$$u_0^{k+1} = a(t_{k+1}), \quad u_n^{k+1} = b(t_{k+1}), \quad (80)$$

for $k = 0, 1, \ldots, l-1$ and

$$u_i^0 = \varphi(x_i), \quad i = 0, \ldots, n. \quad (81)$$

Local truncation error: $O(h^2 + \tau)$. Unconditionally stable.

Cranck-Nicolson scheme:

$$
\begin{aligned}
&-\frac{p\tau}{2h^2}u_{i-1}^{k+1}+\left(1+\frac{p\tau}{h^2}\right)u_i^{k+1}-\frac{p\tau}{2h^2}u_{i+1}^{k+1}= \\
&=\frac{p\tau}{2h^2}u_{i-1}^{k}+\left(1-\frac{p\tau}{h^2}\right)u_i^{k}+\frac{p\tau}{2h^2}u_{i+1}^{k}+\frac{\tau}{2}[f(x_i,t_k)+f(x_i,t_{k+1})],
\end{aligned}
\tag{82}
$$

$$i=1,\ldots,n-1$$

$$u_0^{k+1}=a(t_{k+1}),\quad u_n^{k+1}=b(t_{k+1}),\tag{83}$$

for $k=0,1,\ldots,l-1$ and

$$u_i^0=\varphi(x_i),\quad i=0,\ldots,n.\tag{84}$$

Local truncation error: $O(h^2+\tau^2)$. Unconditionally stable.

Discretization of wave equation

$$u_{tt}(x,t) - pu_{xx}(x,t) = f(x,t), \quad x \in (0,L), \ \ t \in (0,T),$$

$$u(0,t) = a(t), \quad u(L,t) = b(t), \quad t \in (0,T),$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in [0,L].$$

Consistency conditions: $\varphi(0) = a(0)$, $\varphi(L) = b(0)$, $\psi(0) = a'(0)$, $\psi(L) = b'(0)$.

Method of lines:

$x_i = ih$, $h = L/n$.

$\widehat{u}_i(t) \approx u(x_i,t)$

$$\left.\begin{aligned}
&\widehat{u}_i{}''(t) = p\,\frac{\widehat{u}_{i-1}(t) + \widehat{u}_{i+1}(t) - 2\widehat{u}_i(t)}{h^2} + f(x_i,t), \quad t \in (0,T),\\
&\qquad i = 1,\ldots,n-1,\\
&\widehat{u}_0(t) = a(t), \ \ \widehat{u}_n(t) = b(t), \quad t \in (0,T),\\
&\widehat{u}_i(0) = \varphi(x_i), \quad \widehat{u}_i{}'(0) = \psi(x_i), \quad i = 0,\ldots,n.
\end{aligned}\right\} \quad (85)$$

Local truncation error: $O(h^2)$.

Explicit scheme:

$$u_i^{k+1} = 2\left(1 - \frac{p\tau^2}{h^2}\right)u_i^k + \frac{p\tau^2}{h^2}\left(u_{i-1}^k + u_{i+1}^k\right) - u_i^{k-1} + \tau^2 f(x_i, t_k),$$

$$i = 1, \ldots, n-1$$

$$u_0^{k+1} = a(t_{k+1}), \quad u_n^{k+1} = b(t_{k+1}),$$

for $k = 0, 1, \ldots, l-1$ and

$$u_i^0 = \varphi(x_i), \quad i = 0, \ldots, n,$$

$$u_i^1 = u_i^0 + \tau\psi(x_i) + \frac{\tau^2}{2}\left(p\varphi''(x_i) + f(x_i, 0)\right), \quad i = 0, \ldots, n.$$

Here $u_i^k \approx \widehat{u}_i(t_k)$.

Local truncation error: $O(h^2 + \tau^2)$. Conditionally stable. Stability condition:

$$\frac{\sqrt{p}\tau}{h} \leq 1.$$