Numerical Methods II. Homework Problems

1. Solve the problem

$$-3u''(x) + (x+2)u(x) = 4x, \quad x \in (-1,1),$$
$$u'(-1) + 4u(-1) = 3, \quad -u'(1) + 2u(1) = 0$$

   by means of the finite difference method with stepsize $h = 0.001$. Plot the graph.

2. Solve the problem

$$-u''(x) + 2xu(x) \;=\; x^2\,,\; x \in (1,2)\,,$$
$$u'(1) = u(2) = 0,$$

   by means of the Galerkin finite element method with shape functions

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere,} \end{cases}$$

   where $x_i = 1 + ih$, $h = 0.05$. Plot the graph.

3. Solve the problem

$$-u''(x) + (4-x)u(x) \;=\; x + 5\,,\; x \in (0,2)\,,$$
$$u'(0) = 1,\; u'(2) = 2,$$

   by means of the Galerkin finite element method with shape functions

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere,} \end{cases}$$

   where $x_i = ih$, $h = 0.01$. Plot the graph.

4. Solve the problem

$$-u''(x) + 6u(x) \;=\; -\sqrt{x}\,,\; x \in (0,1)\,,$$
$$u(0) = u(1) = 0$$

by means of finite volume method with approximation $u_h = \sum_{j=0}^{n-1} u_i \varphi_i$,

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere} \end{cases}$$

on the mesh $\mathcal{M} = \bigcup_{j=1}^{n-1} \omega_j$, where $\omega_j = (x_{j-1/2}, x_{j+1/2})$, $j = 1, \ldots, n-1$, $x_j = jh$, $j = 1, \ldots, n$, $h = 0.05$ and $n = 20$. Plot the graph.

5. Solve the problem

$$\begin{aligned} -((x^2+1)u')'(x) &= x\,, \ x \in (0,1)\,, \\ u(0) = 2,\ u(1) = 1 \end{aligned}$$

by means of finite volume method with approximation $u_h = \sum_{j=0}^{n-1} u_i \varphi_i$,

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere} \end{cases}$$

on the mesh $\mathcal{M} = \bigcup_{j=1}^{n-1} \omega_j$, where $\omega_j = (x_{j-1/2}, x_{j+1/2})$, $j = 1, \ldots, n-1$, $x_j = jh$, $j = 1, \ldots, n$, $h = 0.05$ and $n = 20$. Plot the graph.

6. Solve the problem

$$\begin{aligned} &-\Delta u(x,y) + 7u(x,y) = x^2 - y^2, \ \ (x,y) \in \Omega, \\ &u(x,y) = \cos y, \ \ (x,y) \in \Gamma, \end{aligned}$$

where $\Omega = \{(x,y) \,:\, 1 \le x \le 3,\ 1 \le y \le 4\}$ and Γ is the boundary of Ω, by means of the finite difference method with stepsizes $h_x = h_y = 0.02$. Plot the graph.

7. Solve the problem

$$\begin{aligned} &-\Delta u(x,y) + 1.3u(x,y) = x^2 - y^2, \ \ (x,y) \in \Omega, \\ &u(x,y) = xy - 5, \ \ (x,y) \in \Gamma, \end{aligned}$$

where $\Omega = \{(x,y) \,:\, -1 \le x \le 0,\ 1 \le y \le 2\}$ and Γ is the boundary of Ω, by means of the finite difference method with stepsizes $h_x = h_y = 0.01$. Plot the graph.

8. Solve the problem

$$-3u_{xx}(x,y) - 4u_{yy}(x,y) + (x^2+2y^2)u(x,y) = 7(x-y),$$
$$x \in (0,2), \quad y \in (0,1),$$

$$u(x,0) = 5x-3,\ u(x,1) = -2x-3,\ u(0,y) = -3,\ -yu_x(2,y) + u(2,y) = 7$$

by means of the finite difference method with stepsizes $h_x = h_y = 0.01$. Plot the graph.

9. Solve the problem

$$u_t(x,t) - 4u_{xx}(x,t) = \cos(x+t)\,,\ x \in (-1,1)\,, t \in (0,2),$$
$$u(-1,t) = u(1,t) = 1\,,\ t \in (0,2),$$
$$u(x,0) = 1 + x^2\,,\ x \in [-1,1]$$

by means of Cranck-Nicolson scheme with stepsizes $h = \tau = 0.01$. Plot the graph.

10. Construct an explicit difference scheme of 2nd order local accuracy for the problem

$$u_{tt}(x,t) + u_t(x,t) - u_{xx}(x,t) - u_x(x,t) = x + t\,,\ x \in (0,1)\,, t \in (0,4),$$
$$u(0,t) = u(1,t) = 0\,,\ t \in (0,4),$$
$$u(x,0) = u_t(x,0) = 0\,,\ x \in [0,1].$$

and solve the problem with the space stepsize $h = 0.01$ and different time stepsizes τ. Numerically, find maximum τ for maintaining the stability.