Method of integral transforms

Laplace transform

This transform can be used to replace the time derivative by multiplication with a certain variable.

Laplace transform of a function $u(t)$, $t > 0$, is defined by

$$(\mathcal{L}u)(s) = U(s) = \int_0^\infty u(t)e^{-st}dt$$

where s is a complex number.

If u is exponentially bounded, i.e.

$$|u(t)| \leq Ce^{\sigma t}, \quad t > 0, \quad C, \sigma - \text{some constants}$$

then its Laplace transform exists for variables s belonging to right half-plane $\operatorname{Re} s > \sigma$, $\sigma \in \mathbb{R}$.

Properties of Laplace transform:

$$\mathcal{L}(c_1 u + c_2 v) = c_1 \mathcal{L}u + c_2 \mathcal{L}v\,, \quad c_1, c_2 \text{ - constants}$$

$$(\mathcal{L}u')(s) = sU(s) - u(0)$$

$$(\mathcal{L}u'')(s) = s^2 U(s) - su(0) - u'(0)$$

For the time convolution

$$(u * v)(t) = \int_0^t u(t-\tau)v(\tau)d\tau$$

the following formula is valid:

$$\mathcal{L}(u * v)(s) = (\mathcal{L}u)(s)\,(\mathcal{L}v)(s) = U(s)V(s)$$

The procedure of solving Cauchy and initial boundary value problems by means of Laplace transform:

1. Apply the Laplace transform $\mathcal{L}$ to the problem.
2. Solve the transformed problem.
3. Apply the inverse Laplace transform $\mathcal{L}^{-1}$ to the solution.

Transformation of t- and x-dependent function $u(x,t)$ and its derivatives:

$$(\mathcal{L}u)(x,s) = U(x,s) = \int_0^\infty u(x,t)e^{-st}dt$$

$$(\mathcal{L}u_t)(x,s) = sU(x,s) - u(x,0)$$

$$(\mathcal{L}u_{tt})(x,s) = s^2U(x,s) - su(x,0) - u_t(x,0)$$

$$(\mathcal{L}u_{xx})(x,s) = \int_0^\infty u_{xx}(x,t)e^{-st}dt = \frac{\partial^2}{\partial x^2}\int_0^\infty u(x,t)e^{-st}dt = U_{xx}(x,s)$$

If u is bounded for $t > 0$, $x > 0$ then

$U(x, s)$ is bounded for $s \in \mathbb{R}$, $s > 1$, $x > 0$.

Indeed:

$$|u(x,t)| \leq C,\ C \text{ - constant} \quad \Rightarrow$$

$$|U(x,s)| = \left| \int_0^{\infty} u(x,t) e^{-st} dt \right| \leq \int_0^{\infty} |u(x,t)| e^{-st} dt \leq$$

$$\leq \int_0^{\infty} C e^{-st} dt = C \int_0^{\infty} e^{-st} dt = \frac{C}{s} \leq C \ \text{ if } s > 1.$$

Fourier transform

This transform can be used to replace the space derivative by multiplication with a certain variable.

Fourier transform of a function $u(x)$, $x \in \mathbb{R}$, is defined by

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) = \int_{-\infty}^{\infty} u(x) e^{-\mathrm{i}\xi x} dx$$

where ξ is a real number.

If u is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |u(x)| dx < \infty$$

then its Fourier transform exists for $\xi \in \mathbb{R}$.

Properties of Fourier transform:

$$\mathcal{F}(c_1 u + c_2 v) = c_1 \mathcal{F} u + c_2 \mathcal{F} v \,, \quad c_1, c_2 \text{ - constants}$$

$$(\mathcal{F} u')(\xi) = \mathrm{i}\xi \hat{u}(\xi)$$

$$(\mathcal{F} u'')(\xi) = (\mathrm{i}\xi)^2 \hat{u}(\xi) = -\xi^2 \hat{u}(\xi)$$

For the convolution

$$(u * v)(x) = \int_{-\infty}^{\infty} u(x-y) v(y) dy$$

the following formula is valid:

$$\mathcal{F}(u * v)(\xi) = (\mathcal{F} u)(\xi)\,(\mathcal{F} v)(\xi) = \hat{u}(\xi)\hat{v}(\xi)$$

Formula for inverse Fourier transform

$$(\mathcal{F}^{-1} \hat{u})(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\xi) e^{\mathrm{i}\xi x} d\xi$$

The procedure of solving Cauchy problems by means of Fourier transform:

1. Apply the Fourier transform $\mathcal{F}$ to the problem.
2. Solve the transformed problem.
3. Apply the inverse Fourier transform $\mathcal{F}^{-1}$ to the solution.

Transformation of x- and t-dependent function $u(x,t)$ and its derivatives:

$$(\mathcal{F}u)(\xi,t) = \hat{u}(\xi,t) = \int_{-\infty}^{\infty} u(x,t)e^{-\mathrm{i}\xi x}dx$$

$$(\mathcal{F}u_{xx})(\xi,t) = -\xi^2\hat{u}(\xi,t)$$

$$(\mathcal{F}u_t)(\xi,t) = \int_{-\infty}^{\infty} u_t(x,t)e^{-\mathrm{i}\xi x}dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)e^{-\mathrm{i}\xi x}dx = \hat{u}_t(\xi,t)$$

$$(\mathcal{F}u_{tt})(\xi,t) = \int_{-\infty}^{\infty} u_{tt}(x,t)e^{-\mathrm{i}\xi x}dx = \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} u(x,t)e^{-\mathrm{i}\xi x}dx = \hat{u}_{tt}(\xi,t)$$