Some more general mathematics.

Eigenvalues and eigenfunctions of operators.

Let L be a linear operator.

A number λ is called an *eigenvalue* of L if there exists a nontrivial (i.e. not identically zero) function v such that

$$Lv = \lambda v$$

The function v is called an *eigenfunction* that corresponds to the eigenvalue λ.

Examples.

1. Eigenvalues of the operator -$\frac{d^2}{dx^2}$ subject to homogeneous Dirichlet boundary conditions.

We search for nontrivial functions $v(x)$ satisfying

$$-\frac{d^2}{dx^2}\,v(x) = \lambda\, v(x), \quad 0 < x < l, \quad v(0) = v(l) = 0.$$

or equivalently,

$$v''(x) + \lambda\, v(x) = 0, \quad 0 < x < l, \quad v(0) = v(l) = 0.$$

The characteristic equation

$$\mu^2 + \lambda = 0 \quad \Rightarrow \quad \mu = \pm\sqrt{-\lambda}$$

Three cases:

a) $\lambda < 0$. Then the general solution of the equation $v'' + \lambda v = 0$ is

$$v(x) = Ce^{-\sqrt{\omega}x} + De^{\sqrt{\omega}x}\,, \quad \omega = -\lambda\,, \quad C, D \in \mathbb{R}$$

Boundary conditions yield the linear homogeneous system of equations

$$\begin{aligned} &C + D = 0 \\ &e^{-\sqrt{\omega}l}C + e^{\sqrt{\omega}l}D = 0 \end{aligned}$$

for the constants C and D. It has a regular matrix (determinant differs from zero), hence the solution is trivial, i.e. $C = D = 0$.

b) $\lambda = 0$. Then the general solution of the equation $v'' + \lambda v = 0$ is

$$v(x) = C + Dx\,, \quad C, D \in \mathbb{R}$$

Boundary conditions yield the linear homogeneous system of equations

$$C = 0$$
$$C + lD = 0$$

that again has the trivial solution $C = D = 0$.

c) $\lambda > 0$. Then the general solution of the equation $v'' + \lambda v = 0$ is

$$v(x) = C \sin \sqrt{\lambda} x + D \cos \sqrt{\lambda} x \,, \quad C, D \in \mathbb{R}$$

First coundary condition yields $v(0) = C \sin 0 + D \cos 0 = 0 \Rightarrow D = 0$. Thus $v(x) = C \sin \sqrt{\lambda} x$.

Second boundary condition gives

$$v(l) = C \sin \sqrt{\lambda} l = 0.$$

It holds $\sin z = 0$ for $z = n\pi$, $n = 1, 2, \ldots$. Therefore,

$$\sin \sqrt{\lambda} l = 0 \text{ for } \sqrt{\lambda} l = n\pi \quad \Rightarrow \quad \lambda = \left(\frac{n\pi}{l}\right)^2 , \quad n = 1, 2, \ldots$$

In such a case, constant C may be arbitrary and the problem

$$v''(x) + \lambda \, v(x) = 0, \quad 0 < x < l, \quad v(0) = v(l) = 0.$$

has a nontrivial solution $v(x) = C \sin \sqrt{\lambda} x$.

We see that the problem

$$v''(x) + \lambda\, v(x) = 0, \quad 0 < x < l, \quad v(0) = v(l) = 0$$

has nontrivial solutions only in case

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, \ldots$$

and they are

$$v = v_n = C_n \sin\frac{n\pi x}{l}, \quad n = 1, 2, \ldots$$

where C_n are arbitrary constants.

Eigenvalues and the corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \;\; v_n = \sin\frac{n\pi x}{l}, \quad n = 1, 2, \ldots$$

2. Eigenvalues of the operator -$\frac{d^2}{dx^2}$ subject to homogeneous Neumann boundary conditions.

We search for nontrivial functions $v(x)$ satisfying

$$-\frac{d^2}{dx^2}\,v(x) = \lambda\,v(x), \quad 0 < x < l, \quad v'(0) = v'(l) = 0.$$

or equivalently,

$$v''(x) + \lambda\,v(x) = 0, \quad 0 < x < l, \quad v'(0) = v'(l) = 0.$$

Nontrivial solutions exist only in case

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 0, 1, 2, \ldots$$

and they are

$$v = v_n = C_n \cos\frac{n\pi x}{l}, \quad n = 0, 1, 2, \ldots$$

where C_n are arbitrary constants.

Eigenvalues and the corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \;\; v_n = \cos\frac{n\pi x}{l}, \quad n = 0, 1, 2, \ldots$$

Fourier series. Fourier sine and cosine series.

A $2l$ - periodic function $F(x)$ can be expanded into *Fourier series*

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right]$$

where

$$a_0 = \frac{1}{l} \int_{-l}^{l} F(x) dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} F(x) \cos \frac{n\pi x}{l} dx \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{l} \int_{-l}^{l} F(x) \sin \frac{n\pi x}{l} dx \quad n = 1, 2, \ldots$$

In case the $2l$-periodic function $F(x)$ is *odd* then

$$a_n = 0, \quad n = 0, 1, 2, \ldots$$

and Fourier series collapses to *Fourier sine series*

$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where

$$b_n \;=\; \frac{1}{l} \int_{-l}^{l} F(x) \sin \frac{n\pi x}{l} dx \;=\; \frac{2}{l} \int_{0}^{l} F(x) \sin \frac{n\pi x}{l} dx \quad n = 1, 2, \ldots$$

We point out that Fourier sine series contains eigenfunctions of the operator -$\frac{d^2}{dx^2}$ subject to homogeneous Dirichlet boundary conditions

In case the $2l$-periodic function $F(x)$ is *even* then

$$b_n = 0, \quad n = 1, 2, \dots$$

and Fourier series collapses to *Fourier cosine series*

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where

$$a_0 = \frac{1}{l} \int_{-l}^{l} F(x)dx = \frac{2}{l} \int_{0}^{l} F(x)dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} F(x) \cos \frac{n\pi x}{l} dx = \frac{2}{l} \int_{0}^{l} F(x) \cos \frac{n\pi x}{l} dx \quad n = 1, 2, \dots$$

We point out that Fourier cosine series contains eigenfunctions of the operator -$\frac{d^2}{dx^2}$ subject to homogeneous Neumann boundary conditions

Initial boundary value problems on finite interval $0 < x < l$.

Problem for homogeneous wave equation with homogeneous Dirichlet boundary conditions

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0\,, \quad 0 < x < l,\ t > 0$$
$$u(0,t) = u(l,t) = 0\,, \quad t > 0$$
$$u(x,0) = \varphi(x)\,, \quad u_t(x,0) = \psi(x)\,, \quad 0 < x < l$$

Separation of variables. Suppose that the solution has the form

$$u(x,t) = X(x)T(t)$$

Let us insert this solution into the equation and separate t-dependent function T and x-dependent function X:

$$X(x)T''(t) - c^2X''(x)T(t) = 0 \;\Rightarrow$$

$$X(x)T''(t) = c^2X''(x)T(t) \;\Rightarrow$$

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)}$$

Right-hand side of this equality is independent of t. This means that $\frac{T''(t)}{c^2T(t)}$ is constant. Similarly, the left-hand side is independent of x. This means that $\frac{X''(x)}{X(x)}$ is constant.

Consequently,

$$-\frac{T''(t)}{c^2T(t)} = -\frac{X''(x)}{X(x)} = \lambda$$

where λ is a constant.

We obtain 2 equations

$$X'' + \lambda X = 0$$
$$T'' + c^2 \lambda T = 0$$

Let us consider the first equation with given homogeneous Dirichlet boundary conditions:

$$X''(x) + \lambda X(x) = 0\,, \quad 0 < x < l\,, \qquad X(0) = X(l) = 0$$

This is the *eigenvalue problem* for the operator $-\frac{d^2}{dx^2}$.

Nontrivial solutions $X(x) \not\equiv 0$ exist only in case

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2\,, \quad n = 1, 2, \dots$$

and they are

$$X(x) = X_n(x) = D_n \sin \frac{n\pi x}{l}$$

where D_n is are arbitrary constants.

Let us consider the second equation

$$T''(t) + c^2\lambda T(t) = 0$$

The general solution corresponding to $\lambda = \lambda_n$ is

$$T(t) = T_n(t) = A_n \cos\sqrt{c^2\lambda_n}t + B_n \sin\sqrt{c^2\lambda_n}t =$$

$$= A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}$$

where A_n and B_n are arbitrary constants.

We have deduced the following family of solutions of the equation $u_{tt} - c^2 u_{xx} = 0$ that satisfy the homogeneous Dirichlet boundary conditions:

$$u(x,t) = u_n(x,t) = X_n(x)T_n(t) =$$

$$\left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}, \quad n = 1, 2, \dots$$

where A_n and B_n are arbitrary constants.

Since the equation is linear, the following series

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

is also the solution of the equation $u_{tt} - c^2 u_{xx} = 0$.

It satisfies the homogeneous Dirichlet boundary conditions, too.

This is a Fourier *sine series* with respect to x.

Setting $t = 0$ in the formulas of u and u_t we have

$$\varphi(x) = u(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

$$\psi(x) = u_t(x, 0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$$

Consequently, A_n and $B_n \frac{n\pi c}{l}$ are the Fourier sine series coefficients of φ and ψ, respectively.

Hence, the formulas of the constants A_n and B_n are

$$\begin{aligned} A_n &= \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx \quad n = 1, 2, \ldots \\ B_n &= \frac{2}{n\pi c} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx \quad n = 1, 2, \ldots \end{aligned}$$

The coefficients in front of t in the formula of $u(x, t)$, i.e.

$$\frac{n\pi c}{l}$$

are called *frequencies*.

Problem for homogeneous diffusion equation
with homogeneous Dirichlet boundary conditions

$$\begin{aligned} &u_t(x,t) - k u_{xx}(x,t) = 0\,, \quad 0 < x < l,\ t > 0 \\ &u(0,t) = u(l,t) = 0\,, \quad t > 0 \\ &u(x,0) = \varphi(x)\,, \quad 0 < x < l \end{aligned}$$

Separation of variables. Suppose that

$$u(x,t) = X(x)T(t)$$

Then

$$-\frac{T'(t)}{kT(t)} = -\frac{X''(x)}{X(x)} = \lambda$$

where λ is an unknown constant.

We obtain 2 equations

$$X'' + \lambda X = 0$$
$$T' + k\lambda T = 0$$

The first equation with given homogeneous Dirichlet boundary conditions:

$$X''(x) + \lambda X(x) = 0\,, \quad 0 < x < l\,, \qquad X(0) = X(l) = 0$$

Nontrivial solutions exist only for the eigenvalues:

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2\,, \quad n = 1, 2, \ldots$$

and the general solution corresponding to λ_n is

$$X(x) = X_n(x) = D_n \sin\frac{n\pi x}{l}$$

where D_n is an arbitrary constant.

Let us consider the second equation

$$T'(t) + k\lambda T(t) = 0$$

The general solution corresponding to $\lambda = \lambda_n$ is

$$T(t) = T_n(t) = A_n e^{-(\frac{n\pi}{l})^2 kt}$$

where A_n is an arbitrary constant.

We have obtained the following family of solutions of the equation $u_t - ku_{xx} = 0$ that satisfy the homogeneous Dirichlet boundary conditions:

$$u(x,t) = u_n(x,t) = X_n(x)T_n(t) =$$

$$A_n e^{-(\frac{n\pi}{l})^2 kt} \sin \frac{n\pi x}{l}, \quad n = 1, 2, \ldots$$

where A_n are arbitrary constants.

Since the equation is linear, the following series

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \sin \frac{n\pi x}{l}$$

is also the solution of the equation $u_t - ku_{xx} = 0$.

It satisfies the homogeneous Dirichlet boundary conditions, too.

This is again a Fourier sine series with respect to x.

Setting $t = 0$ in the formula of u we have

$$\varphi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

Consequently, A_n are the Fourier sine series coefficients of φ.

The formulas of the constants A_n are

$$A_n \;=\; \frac{2}{l}\int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx \quad n = 1, 2, \ldots$$

Problem for homogeneous diffusion equation with homogeneous Neumann boundary conditions

$$\begin{aligned} &u_t(x,t) - ku_{xx}(x,t) = 0\,, \quad 0 < x < l,\ t > 0 \\ &u_x(0,t) = u_x(l,t) = 0\,, \quad t > 0 \\ &u(x,0) = \varphi(x)\,, \quad 0 < x < l \end{aligned}$$

Separation of variables. Suppose again that

$$u(x,t) = X(x)T(t)$$

Then, as before,

$$-\frac{T'(t)}{kT(t)} = -\frac{X''(x)}{X(x)} = \lambda$$

where λ is an unknown constant and

$$\begin{aligned} X'' + \lambda X = 0 \\ T' + k\lambda T = 0 \end{aligned}$$

This time we have to solve the first equation with given homogeneous Neumann boundary conditions:

$$X''(x) + \lambda X(x) = 0\,, \quad 0 < x < l\,, \qquad X'(0) = X'(l) = 0$$

Nontrivial solutions exist only in case of the eigenvalues:

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2\,, \quad n = 0, 1, 2, \ldots$$

(NB! Unlike the case of Dirichlet boundary conditions, the value $\lambda_0 = 0$ is now also included!)

The general solution corresponding to λ_n is

$$X(x) = X_n(x) = C_n \cos\frac{n\pi x}{l}$$

where C_n is an arbitrary constant.

The general solution of the second equation

$$T'(t) + k\lambda T(t) = 0$$

corresponding to $\lambda = \lambda_n$ is again

$$T(t) = T_n(t) = A_n e^{-(\frac{n\pi}{l})^2 kt}$$

where A_n is an arbitrary constant.

We have obtained the following family of solutions of the equation $u_t - ku_{xx} = 0$ that satisfy the homogeneous Neumann boundary conditions:

$$u(x,t) = u_n(x,t) = X_n(x)T_n(t) =$$

$$A_n e^{-(\frac{n\pi}{l})^2 kt} \cos \frac{n\pi x}{l}\,, \quad n = 0, 1, 2, \ldots$$

where A_n are arbitrary constants.

The following series

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \cos \frac{n\pi x}{l}$$

is also a solution of the equation $u_t - ku_{xx} = 0$.

It satisfies the homogeneous Neumann boundary conditions, too.

This is a Fourier *cosine series* with respect to x.

Setting $t = 0$ in the formula of u we have

$$\varphi(x) = u(x,0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}$$

Thus, A_n are the Fourier cosine series coefficients of φ.

The formulas of the constants A_n are

$$A_0 = \frac{2}{l} \int_0^l \varphi(x) dx$$

$$A_n = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{n\pi x}{l} dx \quad n = 1, 2, \ldots$$

Problems for homogeneous equations
with homogeneous Robin boundary conditions

Example: A problem for diffusion equation with mixed Dirichlet and Robin boundary conditions:

$$
\begin{aligned}
&u_t(x,t) - k u_{xx}(x,t) = 0\,, \quad 0 < x < 1,\ t > 0 \\
&u(0,t) = 0, \quad t > 0 \\
&u_x(1,t) + h u(1,t) = 0\,, \quad t > 0 \\
&u(x,0) = \varphi(x)\,, \quad 0 < x < 1
\end{aligned}
$$

where $h > 0$.

Separation of variables. Suppose

$$u(x,t) = X(x)T(t)$$

This yields again

$$-\frac{T'(t)}{kT(t)} = -\frac{X''(x)}{X(x)} = \lambda$$

where λ is an unknown constant, and

$$X'' + \lambda X = 0$$
$$T' + k\lambda T = 0$$

We have to solve the following eigenvalue problem for the first equation:

$$X''(x) + \lambda X(x) = 0\,, \quad 0 < x < 1\,,$$
$$X(0) = 0, \quad X'(1) + hX(1) = 0$$

The characteristic equation

$$\mu^2 + \lambda = 0 \quad \Rightarrow \quad \mu = \pm\sqrt{-\lambda}$$

Three cases:

a) $\lambda < 0$. Then the general solution of the equation $X'' + \lambda X = 0$ is

$$X(x) = Ce^{-\sqrt{\omega}x} + De^{\sqrt{\omega}x}\,, \quad \omega = -\lambda\,, \quad C, D \in \mathbb{R}$$

Boundary conditions yield the linear homogeneous system of equations

$$C + D = 0$$
$$(-\sqrt{\omega} + h)e^{-\sqrt{\omega}}C + (\sqrt{\omega} + h)e^{\sqrt{\omega}}D = 0$$

for the constants C and D. It has a regular matrix (determinant differs from zero), hence the solution is trivial, i.e. $C = D = 0$.

b) $\lambda = 0$. Then the general solution of the equation $X'' + \lambda X = 0$ is

$$X(x) = C + Dx\,, \quad C, D \in \mathbb{R}$$

Boundary conditions yield the linear homogeneous system of equations

$$\begin{aligned} &C = 0 \\ &hC + (1+h)D = 0 \end{aligned}$$

that again has the trivial solution $C = D = 0$.

c) $\lambda > 0$. Then the general solution of the equation $X'' + \lambda X = 0$ is

$$X(x) = C \sin \sqrt{\lambda} x + D \cos \sqrt{\lambda} x\,, \quad C, D \in \mathbb{R}$$

First coundary condition yields $X(0) = C \sin 0 + D \cos 0 = 0 \;\Rightarrow\; D = 0$. Thus $X(x) = C \sin \sqrt{\lambda} x$.

Second boundary condition gives

$$X'(1) + hX(1) = C(\sqrt{\lambda} \cos \sqrt{\lambda} + h \sin \sqrt{\lambda}) = 0.$$

Thus,

$$C\left(\tan \sqrt{\lambda} + \frac{\sqrt{\lambda}}{h}\right) = 0.$$

Denote $\mu = \sqrt{\lambda}$. Then the equation is

$$C\left(\tan \mu + \frac{\mu}{h}\right) = 0.$$

The equation $\tan \mu = -\frac{\mu}{h}$ has infinitely many positive solutions $\mu_1 < \mu_2 \ldots$.

Summing up, eigenvalues: $\lambda_n = \mu_n^2$ where μ_n, $n = 1, 2, \ldots$, are positive solutions of the equation

$$\tan \mu = -\frac{\mu}{h}$$

and the corresponding eigenfunctions are

$$X_n(x) = \sin \mu_n x, \quad n = 1, 2, \ldots$$

The general solution of the second equation

$$T'(t) + k\lambda T(t) = 0$$

corresponding to $\lambda = \lambda_n = \mu_n^2$ is

$$T(t) = T_n(t) = A_n e^{-\mu_n^2 kt}$$

where A_n is an arbitrary constant.

We construct the solution of the equation $u_t - ku_{xx} = 0$ satisfying the homogeneous boundary conditions in the form of series

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\mu_n^2 kt} \sin \mu_n x$$

Taking $t = 0$ we have

$$\varphi(x) = \sum_{i=1}^{\infty} A_n \sin \mu_n x$$

The system of functions $\sin \mu_n x$, $n = 1, 2, \ldots$ is orthogonal:

$$\int_0^1 \sin \mu_n x \sin \mu_m x dx = \begin{cases} 0 & \text{if } \; n \neq m \\ \frac{\mu_n - \sin \mu_n \cos \mu_n}{2\mu_n} & \text{if } \; n = m \end{cases}$$

We obtain the following formulas for the coefficients:

$$A_n = \frac{2\mu_n}{\mu_n - \sin \mu_n \cos \mu_n} \int_0^1 \varphi(x) \sin \mu_n x dx\,, \quad n = 1, 2, \ldots$$

Diffusion equation with two Robin boundary conditions:

$$
\begin{aligned}
&u_t(x,t) - k u_{xx}(x,t) = 0\,, \quad 0 < x < l,\ t > 0 \\
&\alpha_0 u(0,t) + \beta_0 u_x(0,t) = 0, \quad t > 0 \\
&\alpha_1 u(l,t) + \beta_1 u_x(l,t) = 0, \quad t > 0 \\
&u(x,0) = \varphi(x)\,, \quad 0 < x < l
\end{aligned}
$$

where $\alpha_0^2 + \beta_0^2 \neq 0$, $\alpha_1^2 + \beta_1^2 \neq 0$.

The problem for X is:

$$
\begin{aligned}
&X''(x) + \lambda X(x) = 0\,, \quad 0 < x < l\,, \\
&\alpha_0 X(0) + \beta_0 X'(0) = 0, \\
&\alpha_1 X(l) + \beta_1 X'(l) = 0
\end{aligned}
$$

This is the *Sturm-Liouville boundary value problem.*

It is possible to prove that

1) All eigenvalues real, and form an increasing infinite sequence

$$
\lambda_1 < \lambda_2 < \dots
$$

2) Eigenfunctions X_n and X_m corresponding to different eigenvalues λ_n and λ_m are orthogonal.

This enables again to represent the solution in the form of series

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

Problems for nonhomogeneous equations with homogeneous boundary conditions

Method is as follows:

We find the system of eigenfunctions X_n as in the case of the homogeneous equation,

expand the solution u and the right-hand side f into series with respect to X_n:

$$u(x,t) = \sum_{i=1}^{\infty} T_n(t) X_n(x)$$

$$f(x,t) = \sum_{i=1}^{\infty} f_n(t) X_n(x)$$

construct and solve *nonhomogeneous* equations for T_n.

Example: Problem for diffusion equation with Dirichlet boundary conditions

$$
\begin{aligned}
& u_t(x,t) - k u_{xx}(x,t) \, = f(x,t)\,, \;\; 0 < x < l,\; t > 0 \\
& u(0,t) = u(l,t) = 0\,, \;\; t > 0 \\
& u(x,0) = \varphi(x)\,, \quad 0 < x < l
\end{aligned}
$$

The corresponding eigenvalue problem is

$$
X''(x) + \lambda X(x) = 0\,, \;\; 0 < x < l\,, \qquad X(0) = X(l) = 0
$$

Eigenvalues:

$$
\lambda_n = \left(\frac{n\pi}{l}\right)^2\,, \quad n = 1, 2, \ldots
$$

and eigenfunctions:

$$
X_n(x) = \sin\frac{n\pi x}{l}
$$

The solution is searched in the form of Fourier sine series

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{l}$$

and the right-hand side is also expanded into the sine series

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{l}$$

The coefficients are

$$f_n(t) = \frac{2}{l} \int_0^l f(x,t) \sin \frac{n\pi x}{l} dx.$$

We plug these series into the equation and compute the derivatives:

$$\sum_{n=1}^{\infty} T_n'(t) \sin \frac{n\pi x}{l} + k \sum_{n=1}^{\infty} T_n(t)\lambda_n \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{l}.$$

Thus,

$$\sum_{n=1}^{\infty} \Big[T_n'(t) + k\lambda_n T_n(t) - f_n(t) \Big] \sin \frac{n\pi x}{l} = 0$$

Since the system of functions $\sin \frac{n\pi x}{l}$ is orthogonal, we have

$$T_n'(t) + k\lambda_n T_n(t) = f_n(t) \quad \text{for any } n = 1, 2, \ldots.$$

To complement these ODE-s with initial conditions, we expand $\varphi(x)$, too:

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n \sin \frac{n\pi x}{l}$$

where

$$\varphi_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx.$$

We arrive at the sequence of Cauchy problems for 1st order ODE:

$$T_n'(t)+k\lambda_n T_n(t) = f_n(t), \quad T_n(0) = \varphi_n, \quad n = 1, 2, \dots . \qquad (*)$$

Remember (see Ptk 5) that the solution of the Cauchy problem

$$\frac{d}{dt}v(t) + Av(t) = f(t),\ t > 0, \quad v(0) = \varphi,$$

where A and φ are given numbers, is

$$v(t) = \varphi e^{-At} + \int_0^t e^{-A(t-s)} f(s) ds$$

Therefore, the solutions of (*) are

$$T_n(t) = \varphi_n e^{-k\lambda_n t} + \int_0^t e^{-k\lambda_n(t-\tau)} f_n(\tau) d\tau, \quad n = 1, 2, \dots$$

The solution of the original problem is

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n e^{-k\lambda_n t} \sin\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} \int_0^t e^{-k\lambda_n(t-\tau)} f_n(\tau) d\tau =$$

$$= \sum_{n=1}^{\infty} \varphi_n e^{-k\left(\frac{n\pi}{l}\right)^2 t} \sin\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} \int_0^t e^{-k\left(\frac{n\pi}{l}\right)^2 (t-\tau)} f_n(\tau) d\tau$$

Problems with nonhomogeneous boundary conditions

Typically, such problems are transformed to problems with homogeneous boundary conditions by means of changes of variables.

To this end, a function w satisfying the nonhomogeneous boundary conditions is introduced and the function u is represented as

$$u = w + U$$

where U is the new unknown function.

Example:

$$u_t(x,t) - ku_{xx}(x,t) \ = f(x,t)\,, \quad 0 < x < l,\ t > 0$$
$$u(0,t) = g_0(t), \quad u(l,t) = g_1(t)\,, \quad t > 0$$
$$u(x,0) = \varphi(x)\,, \quad 0 < x < l$$

Define

$$w(x,t) = g_0(t)\left(1 - \frac{x}{l}\right) + g_1(t)\frac{x}{l}.$$

Then the function

$$U = u - w$$

satisfies the problem with homogeneous boundary conditions:

$$U_t(x,t) - kU_{xx}(x,t) \ = f(x,t) - w_t(x,t)\,, \quad 0 < x < l,\ t > 0$$
$$U(0,t) = U(l,t) = 0\,, \quad t > 0$$
$$U(x,0) = \varphi(x) - w(x,0)\,, \quad 0 < x < l$$