Laplace and Poisson equations in two dimensions

Laplace operator in two dimensions

$$\Delta u(x,y) \; = \; u_{xx}(x,y) + u_{yy}(x,y)$$

Laplace equation

$$\Delta u = 0$$

Poisson equation

$$\Delta u = f$$

where $f \not\equiv 0$.

An important property of the Laplace operator:

Laplace operator is *invariant* with respect to any transformation consisting of translations and rotations.

Translation:

$$x' = x + a\,, \quad y' = y + b$$

It holds

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}$$

Hence,

$$u_{xx} + u_{yy} \;=\; u_{x'x'} + u_{y'y'}$$

A rotation of a N-dimensional space is given by an orthonormal matrix:

$$\mathbf{x}' = A\,\mathbf{x}\,, \quad A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \dots \\ \mathbf{v}_N \end{pmatrix}, \quad |\mathbf{v}_i| = 1\,, \quad \mathbf{v}_i \cdot \mathbf{v}_j = 0 \text{ if } i \neq j$$

In 2-dimensional case, a rotation is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $a^2 + b^2 = 1$.

Defining α so that $\tan\alpha = \frac{b}{a}$, from Pythagoras theorem we get

$$a = \cos\alpha\,, \quad b = \sin\alpha$$

Consequently,

$$\begin{aligned} x' &= x\cos\alpha + y\sin\alpha \\ y' &= -x\sin\alpha + y\cos\alpha \end{aligned}$$

We have

$$
\begin{aligned}
u_x &= u_{x'} \cos\alpha - u_{y'} \sin\alpha \\
u_y &= u_{x'} \sin\alpha + u_{y'} \cos\alpha \\
u_{xx} &= u_{x'x'} \cos^2\alpha - 2u_{x'y'} \sin\alpha\cos\alpha + u_{y'y'} \sin^2\alpha \\
u_{yy} &= u_{x'x'} \sin^2\alpha + 2u_{x'y'} \sin\alpha\cos\alpha + u_{y'y'} \cos^2\alpha
\end{aligned}
$$

Consequently,

$$u_{xx} + u_{yy} \;=\; u_{x'x'} + u_{y'y'}$$

Transformation of the Laplace operator to polar coordinates.

$$x = r\cos\theta\,, \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}\,, \qquad \tan\theta = \frac{y}{x}$$

Then

$$\Delta \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \;=\; \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

(I will not ask the derivation of this formula).

Solving Poisson and Laplace equations in radially symmetric case.

Let the solution u and source term f be radially symmetric, i.e. both depend only on r. Then the Poisson equation

$$\Delta u = f$$

has the form

$$u_{rr} + \frac{1}{r} u_r = f(r)$$

This is equivalent to

$$(r u_r)_r = r f(r)$$

Integrating we obtain

$$r u_r = c_1 + \int_0^r s f(s) ds, \qquad u_r = \frac{c_1}{r} + \frac{1}{r} \int_0^r s f(s) ds$$

and the general solution is

$$u = c_1 \ln r + c_2 + \int_0^r \frac{1}{\sigma} \int_0^\sigma s f(s) ds d\sigma$$

where c_1 and c_2 are arbitrary constants.