Wave equation in one spatial variable – Cauchy problem.

General solution of homogeneous equation

Consider the equation

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R},\ t > 0$$

Factorize it:

$$\left(\frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\,\frac{\partial}{\partial x}\right) u = 0$$

Make the change of variables

$$\xi = x + c\,t, \quad \eta = x - c\,t$$

Then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} = c\,\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta},$$

$$\frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x} = -2c\,\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} + c\,\frac{\partial}{\partial x} = 2c\,\frac{\partial}{\partial \xi}$$

The equation is transformed to the form

$$-4c^2\frac{\partial}{\partial \xi \partial \eta} u = 0 \quad \Longrightarrow \quad u_{\xi\eta} = 0$$

Integrate with respect to η:

$$u_\xi = h(\xi)$$

where h is an arbitrary function.

Integrate with respect to ξ:

$$u = \int h(\xi) d\xi + F_2(\eta) \;=\; F_1(\xi) + F_2(\eta)$$

where F_2 is an arbitrary function and $F_1(\xi) = \int h(\xi) d\xi$.

Therefore, we can represent the general solution of the equation $u_{tt} - c\,u_{xx} = 0$ in the form

$$u(x,t) \;=\; F_1(x + c\,t) \;+\; F_2(x - c\,t)$$

where F_1 and F_2 are two arbitrary functions.

In order to guarantee that u is a classical solution of the equation $u_{tt} - c\,u_{xx} = 0$ we must assume that F_1 and F_2 are twice continously differentiable.

Verification.

$$\partial_t^2 u - c^2 \partial_x^2 u = \partial_t^2 [F_1(x + c\,t) \;+\; F_2(x - c\,t)] - c^2 \partial_x^2 [F_1(x + c\,t) \;+\; F_2(x - c\,t)] =$$
$$= c^2 F_1''(x + c\,t) + (-c)^2 F_2''(x + c\,t) - c^2 F_1''(x + c\,t) - c^2 F_2''(x + c\,t) = 0.$$

The term $F_2(x - c\,t)$ represents the wave propagating to the right

The term $F_1(x + c\,t)$ represents the wave propagating to the left

Lines

$$x - c\,t = C, \quad x + c\,t = C, \quad C \in \mathbb{R}$$

are called the *characteristics* of the wave equation $u_{tt} - c\,u_{xx} = 0$.

Cauchy problem for homogeneous equation

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0, \quad x \in \mathbb{R},\ t > 0$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x)\,, \quad x \in \mathbb{R}$$

We will proceed from the general solution

$$u(x,t) \;=\; F_1(x + c\,t) \;+\; F_2(x - c\,t)$$

Setting $t = 0$ we have

$$\varphi(x) \;=\; u(x,0) \;=\; F_1(x) \;+\; F_2(x)$$

Differentiate:

$$\varphi'(x) \;=\; F_1'(x) \;+\; F_2'(x)$$

On the other hand,

$$u_t(x,t) \;=\; c\,F_1'(x + c\,t) \;-\; c\,F_2'(x - c\,t)$$

This yields

$$\psi(x) \;=\; u_t(x,0) \;=\; c\,F_1'(x) \;-\; c\,F_2'(x)$$

We obtain the system for F_1' and F_2':

$$\begin{aligned} F_1'(x) \;+\; F_2'(x) \;&=\; \varphi'(x) \\ c\,F_1'(x) \;-\; c\,F_2'(x) \;&=\; \psi(x) \end{aligned}$$

Solve it:

$$\begin{aligned} F_1'(x) \;&=\; \frac{1}{2}\varphi'(x) + \frac{1}{2c}\psi(x) \\ F_2'(x) \;&=\; \frac{1}{2}\varphi'(x) - \frac{1}{2c}\psi(x) \end{aligned}$$

Integrate:

$$\begin{aligned} F_1(x) \;&=\; \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_0^x \psi(\tau)d\tau + A \\ F_2(x) \;&=\; \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_0^x \psi(\tau)d\tau + B \end{aligned}$$

where A and B are constants.

Taking $x = 0$ and adding we have

$$F_1(0) + F_2(0) = \varphi(0) + A + B$$

Since $F_1(x) + F_2(x) = \varphi(x)$, it holds $A + B = 0$.

Putting the formulas of F_1 and F_2 into the general solution we obtain

$$u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\tau)d\tau$$

This is the *d'Alembert's formula.*

Theorem. Let $\varphi \in C^2$, $\psi \in C^1$. The Cauchy problem

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0, \quad x \in \mathbb{R},\ t > 0$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x)\,, \quad x \in \mathbb{R}$$

has a unique classical solution $u \in C^2$ given by d'Alembert's formula.

Principle of causality.

Initial condition a the point $(x_0, 0)$ can spread only to that part of the xt-plane which lies between the lines with equations $x \pm c\,t = x_0$ (the characteristics passing through the point $(x_0, 0)$).

The sector with these boundary points is called the *domain of influence* of the point $(x_0, 0)$.

Let (x, t) be an arbitrary point. The values of initial conditions on the x-axis (for $t = 0$) influencing the value of the solution at a point (x, t) are:
$\varphi(x - c\,t)$, $\varphi(x + c\,t)$ and
$\psi(\tau)$ for $\tau \in [x - c\,t, x + c\,t]$.

The triangle Δ_{xt} with vertices at the points $(x - c\,t, 0)$, $(x + c\,t, 0)$ and (x, t) is called the *domain of dependence* (or the *characteristic triangle*) of the point (x, t).

Cauchy problem for nonhomogeneous equation

Theorem. Let $\varphi \in C^2$, $\psi \in C^1$, $f \in C^1$. The Cauchy problem

$$\begin{aligned} & u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t), \quad x \in \mathbb{R},\ t > 0 \\ & u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x)\,, \quad x \in \mathbb{R} \end{aligned} \tag{1}$$

has a unique classical solution $u \in C^2$ given by the formula

$$u(x,t) = \frac{1}{2}\left[\varphi(x+c\,t) + \varphi(x-c\,t)\right] + \frac{1}{2c}\int_{x-c\,t}^{x+c\,t} \psi(\tau)d\tau +$$

$$+\frac{1}{2c}\iint\limits_{\Delta_{xt}} f(y,s)dyds \tag{2}$$

Here

$$\iint\limits_{\Delta_{xt}} f(y,s)dyds = \int_0^t \int_{x-c\,(t-s)}^{x+c\,(t-s)} f(y,s)dyds$$

Two methods to derive the solution formula (2).

1. Usage of Green's theorem.

Green's theorem formulated for Δ_{xt}: For any $P, Q \in C^1$,

it holds

$$\iint\limits_{\Delta_{xt}} [P_y - Q_s] dy ds = \int\limits_{\partial\Delta_{xt}} P ds + Q dy$$

where $\int\limits_{\partial\Delta_{xt}}$ is the second kind line integral (counterclockwise).

Denote $\partial\Delta_{xt} = L_0 \cup L_1 \cup L_2$ where

L_0 is the line segment from point $(x - c\,t, 0)$ to point $(x + c\,t, 0)$,

L_1 is the line segment from point $(x + c\,t, 0)$ to point (x, t),

L_2 is the line segment from point (x, t) to point $(x - c\,t, 0)$.

From the differential equation, we have

$$\iint\limits_{\Delta_{xt}} f(y,s)dyds \;=\; \iint\limits_{\Delta_{xt}} [u_{ss} - c^2 u_{yy}]dyds$$

Setting $P = -c^2 u_y$ and $Q = -u_s$ in the Green's theorem we deduce

$$\iint\limits_{\Delta_{xt}} [u_{ss} - c^2 u_{yy}]dyds \;=\; \int\limits_{L_0 \cup L_1 \cup L_2} -c^2 u_y ds - u_s dy$$

On L_0, we have $s = 0$, $ds = 0$ and $u_s(y, 0) = \psi(y)$, thus

$$\int\limits_{L_0} -c^2 u_y ds - u_s dy \; = \; -\int_{x-c\,t}^{x+c\,t} \psi(y) dy$$

On L_1, we have $y + c\,s = x + c\,t$, thus $dy + c\,ds = 0$ and

$$\int\limits_{L_1} -c^2 u_y ds - u_s dy \; = \; c \int\limits_{L_1} u_y dy + u_s ds \; = \; c \int\limits_{L_1} du \; =$$

$$= c\,[u(x,t) - u(x + c\,t, 0)] = c\,u(x,t) - c\,\varphi(x + c\,t)$$

On L_2, we have $y - c\,s = x - c\,t$, thus $dy - c\,ds = 0$ and

$$\int\limits_{L_2} -c^2 u_y ds - u_s dy \; = \; -c \int\limits_{L_2} u_y dy + u_s ds \; = \; -c \int\limits_{L_2} du \; =$$

$$= -c\,[u(x - c\,t, 0) - u(x,t)] = -c\,\varphi(x - c\,t) + c\,u(x,t)$$

Putting pieces together, we obtain

$$\int\limits_{L_0\cup L_1\cup L_2} -c^2 u_y ds - u_s dy \; =$$

$$= 2c\, u(x,t) - c[\varphi(x-c\,t) + \varphi(x+c\,t)] - \int_{x-c\,t}^{x+c\,t} \psi(y)dy$$

and

$$\iint\limits_{\Delta_{xt}} f(y,s)dyds \; =$$

$$= 2c\, u(x,t) - c[\varphi(x-c\,t) + \varphi(x+c\,t)] - \int_{x-c\,t}^{x+c\,t} \psi(y)dy$$

Expressing $u(x,t)$ we obtain the formula

$$u(x,t) = \frac{1}{2}\,[\varphi(x+c\,t) + \varphi(x-c\,t)] \; + \; \frac{1}{2c}\int_{x-c\,t}^{x+c\,t} \psi(\tau)d\tau +$$

$$+\frac{1}{2c}\iint\limits_{\Delta_{xt}} f(y,s)dyds$$

and the formula (2) is derived.

2. Operator method.

Firstly, we solve the following Cauchy problem for ordinary differential equation:

$$\frac{d^2}{dt^2}v(t)+A^2v(t) = f(t),\ t > 0, \quad v(0) = \varphi,\ \frac{d}{dt}v(0) = \psi \qquad (3)$$

where A, φ and ψ are given numbers.

The homogeneous equation is

$$\frac{d^2}{dt^2}v_H(t) + A^2 v_H(t) = 0$$

The characteristic equation $\lambda^2 + A^2 = 0$ has the solutions $\lambda_{12} = \pm A\mathrm{i}$.

The general solution of the homogeneous equation is

$$v_H(t) = C_1 \cos At + C_2 \sin At$$

We use the variation of constants to derive a particular solution of the inhomogeneous equation

$$v_P(t) = D_1(t) \cos At + D_2(t) \sin At$$

Standard system for the derivatives of the coefficients D_1, D_2:

$$\begin{array}{l} D_1' \cos At + D_2' \sin At = 0 \\ D_1' (\cos At)' + D_2' (\sin At)' = f \end{array} \implies \begin{array}{r} D_1' \cos At + D_2' \sin At = 0 \\ -D_1' A \sin At + D_2' A \cos At = f \end{array}$$

Determinant is

$$\text{Det} = \begin{vmatrix} \cos At & \sin At \\ -A \sin At & A \cos At \end{vmatrix} = A(\cos^2 At + \sin^2 At) = A$$

Thus

$$D_1' = \frac{\begin{vmatrix} 0 & \sin At \\ f & A \cos At \end{vmatrix}}{\text{Det}} = \frac{-f(t) \sin At}{A}$$

$$D_2' = \frac{\begin{vmatrix} \cos At & 0 \\ -A \sin At & f \end{vmatrix}}{\text{Det}} = \frac{f(t) \cos At}{A}$$

and

$$D_1(t) = -\frac{1}{A} \int_0^t \sin A\tau f(\tau) d\tau$$

$$D_2(t) = \frac{1}{A} \int_0^t \cos A\tau f(\tau) d\tau$$

The formula of v_P is

$$v_P(t) = D_1(t) \cos At + D_2(t) \sin At =$$

$$= \frac{1}{A} \int_0^t \left[-\sin A\tau \cos At + \cos A\tau \sin At \right] f(\tau) d\tau =$$

$$= \frac{1}{A} \int_0^t \sin A(t - \tau) f(\tau) d\tau$$

The general solution of the equation (3) is

$$v(t) = v_H(t) + v_P(t) = C_1 \cos At + C_2 \sin At + \frac{1}{A}\int_0^t \sin A(t-\tau) f(\tau) d\tau$$

Use the initial conditions $v(0) = \varphi$, $\frac{d}{dt}v(0) = \psi$:

$$v(0) = C_1 \cos 0 + C_2 \sin 0 = C_1 \Longrightarrow C_1 = \varphi$$

$$\frac{d}{dt}v(t) = -C_1 A \sin At + C_2 A \cos At + \int_0^t \cos A(t-\tau) f(\tau) d\tau$$

$$\frac{d}{dt}v(0) = -C_1 A \sin 0 + C_2 A \cos 0 = C_2 A \Longrightarrow C_2 = \frac{\psi}{A}$$

The solution of the Cauchy problem (3) is

$$v(t) \;=\; \varphi \cos At + \psi \frac{1}{A} \sin At + \frac{1}{A}\int_0^t \sin A(t-s) f(s) ds$$

We write this solution in the form

$$v(t) \;=\; \frac{d}{dt}S(t)\varphi + S(t)\psi + \int_0^t S(t-s) f(s) ds$$

where

$$S(t) = \frac{1}{A} \sin At \,, \quad \frac{d}{dt}S(t) = \cos At$$

In the particular case $\varphi = 0,\ f \equiv 0$, the solution is

$$v(t) = S(t)\psi \quad \text{where} \quad S(t) = \frac{1}{A}\sin At$$

Let us compare it with the d'Alembert's formula in case $\varphi = 0$:

$$u(x,t) = \mathcal{S}(t)\psi \ = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\tau)d\tau$$

The operator $\mathcal{S}(t)$ is called *the sine family.*

Using this analogy, we construct the solution of the Cauchy problem for the wave equation in the form

$$u(x,t) = \frac{\partial}{\partial t}\mathcal{S}(t)\varphi + \mathcal{S}(t)\psi + \int_0^t \mathcal{S}(t-s)f(x,s)ds$$

The operator $\frac{\partial}{\partial t}\mathcal{S}(t)$ is called *the cosine family.*

Compute:

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathcal{S}(t)\varphi &= \frac{1}{2c}\,\frac{\partial}{\partial t}\int_{x-c\,t}^{x+c\,t}\varphi(\tau)d\tau = \\
&= \frac{1}{2c}\,\frac{\partial}{\partial t}\left[\int_0^{x+c\,t}\varphi(\tau)d\tau - \int_0^{x-c\,t}\varphi(\tau)d\tau\right] = \\
&= \frac{1}{2c}[c\,\varphi(x+c\,t) - (-c)\varphi(x-c\,t)] = \\
&= \frac{1}{2}[\varphi(x+c\,t) + \varphi(x-c\,t)]
\end{aligned}
$$

$$
\int_0^t \mathcal{S}(t-s)f(x,s)ds = \int_0^t \frac{1}{2c}\int_{x-c\,(t-s)}^{x+c\,(t-s)} f(y,s)dyds =
$$

$$
= \frac{1}{2c}\iint\limits_{\Delta_{xt}} f(y,s)dyds
$$

Let us put the solution together:

$$u(x,t) = \frac{\partial}{\partial t}\mathcal{S}(t)\varphi + \mathcal{S}(t)\psi + \int_0^t \mathcal{S}(t-s)f(x,s)ds =$$

$$= \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\tau)d\tau +$$

$$+\frac{1}{2c}\iint\limits_{\Delta_{xt}} f(y,s)dyds.$$

This is the formula (2).