Linear partial differential equations of the first order

Equation with constant coefficients:

$$a u_x(x,y) + b u_y(x,y) + \lambda u(x,y) = f(x,y)$$

Equation with nonconstant coefficients:

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) + \lambda(x,y)u(x,y) = f(x,y)$$

Firstly, we solve the homogeneous equation with *constant coefficients* and without the term λu

$$au_x(x,y) + bu_y(x,y) = 0$$

Denote

$$\mathbf{v} = (a,b)$$

The derivative of the function u in the direction $\mathbf{v}$ is defined by the formula

$$\frac{\partial}{\partial \mathbf{v}} u = \mathbf{v} \cdot \nabla u = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \right) u$$

Thus, the equation takes the form

$$\frac{\partial}{\partial \mathbf{v}} u(x,y) = 0$$

The function u is constant along lines that are parallel to the vector $\mathbf{v}$.

Equations of a lines parallel to the vector $\mathbf{v}$ are

$$bx - ay = C, \quad C \in \mathbb{R}$$

Such lines are called *characteristics* of the equation $au_x + bu_y = 0$.

The general solution of the equation $au_x + bu_y = 0$ is

$$u(x, y) = F(bx - ay)$$

where F is an arbitrary continuously differentiable function.

Methods to solve the *more general equations with constant coefficients*

$$au_x(x,y) + bu_y(x,y) + \lambda u(x,y) = f(x,y)$$

1. Coordinate method

$$\xi = bx - ay\,, \quad \eta = ax + by$$

Then

$$u_x = bu_\xi + au_\eta, \quad u_y = -au_\xi + bu_\eta,$$
$$au_x + bu_y = a(bu_\xi + au_\eta) + b(-au_\xi + bu_\eta) =$$
$$= (a^2 + b^2)u_\eta$$

The equation $au_x + bu_y + \lambda u = f$ reduces to the following ordinary differential equation:

$$u_\eta + \lambda_1 u = f_1$$

where $\lambda_1 = \frac{\lambda}{a^2+b^2}$, $f_1 = \frac{1}{a^2+b^2} f$.

2. Method of characteristic coordinates

$$\xi = bx - ay\,, \quad \tau = y$$

Then

$$u_x = bu_\xi, \quad u_y = -au_\xi + u_\tau,$$
$$au_x + bu_y = abu_\xi + b(-au_\xi + u_\tau) =$$
$$= bu_\tau$$

The equation $au_x + bu_y + \lambda u = f$ reduces to the following ordinary differential equation:

$$u_\tau + \lambda_1 u = f_1$$

where $\lambda_1 = \frac{\lambda}{b}$, $f_1 = \frac{1}{b} f$.

An alternative change of variables in the method of characteristic coordinates:

$$\xi = bx - ay\,, \quad \tau = x$$

Equation with *nonconstant coefficients.*

Again we start with the homogeneous equation without the term λu

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = 0$$

We introduce the vector field

$$\mathbf{v}(x,y) = (a(x,y), b(x,y))$$

Again,

$$\frac{\partial}{\partial \mathbf{v}} u(x,y) = 0$$

Characteristics are curves whose tangents are the vectors $\mathbf{v}(x,y)$.

Solution u is constant along characteristis.

To determine the characteristics, we have to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

Let the general solution of this equation be

$$h(x,y) = C, \quad C \in \mathbb{R}$$

This is an equation of the characteristics.

The general solution of the equation
$a(x,y)u_x + b(x,y)u_y = 0$ is

$$u(x,y) = F(h(x,y))$$

where F is an arbitrary continuously differentiable function.

Verification.

1. We prove the formula

$$ah_x + bh_y = 0\,.$$

Since $h(x, y) = C$, we have $dh(x, y) = h_x dx + h_y dy = 0$. Using $dy = \frac{b}{a}dx$ we deduce $h_x dx + h_y \frac{b}{a}dx = 0$. Thus, $\frac{dx}{a}(ah_x + bh_y) = 0$. This implies $ah_x + bh_y = 0$.

2. We verify the equation $au_x + bu_y = 0$. Plugging $u = F(h)$ into the left-hand side we obtain $au_x + bu_y = aF'(h)h_x + bF'(h)h_y = F'(h)[ah_x + bh_y] = 0$. This proves that $u = F(h(x, y))$ is the solution of $au_x + bu_y = 0$.

Method of characteristic coordinates for the *general equation with nonconstant coefficients.*

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) + \lambda(x,y)u(x,y) = f(x,y)$$

The change of variables:

$$\xi = h(x,y)\,, \quad \tau = y$$

Then

$$\begin{aligned} &u_x = h_x u_\xi, \quad u_y = h_y u_\xi + u_\tau, \\ &au_x + bu_y = au_\xi h_x + b(u_\xi h_y + u_\tau) = \\ &\ = u_\xi(ah_x + bh_y) + bu_\tau = bu_\tau \end{aligned}$$

The equation $au_x + bu_y + \lambda u = f$ reduces to the following ordinary differential equation with nonconstant coefficients:

$$u_\tau + \lambda_1 u = f_1$$

where $\lambda_1 = \frac{\lambda}{b}$, $f_1 = \frac{1}{b}f$.

An alternative change of variables in the method of characteristic coordinates:

$$\xi = h(x, y)\,, \qquad \tau = x$$

Problems with side conditions.

The solution u is constant along characteristics.

In order to extract a unique particular solution from the general solution, it is sufficient to prescribe the value of u in some point on every characteristic.

Let us introduce a parametric curve:

$$\gamma \;:\quad x = x_0(s),\; y = y_0(s), \quad s \in I,$$

where $I \subset \mathbb{R}$ is some interval.

Let the solution u be prescribed on this curve, i.e.

$$u(x(s), y(s)) = u_0(s), \quad s \in I$$

where u_0 is a given function.

This additional condition determines the solution uniquely in case the curve γ intersects all characteristics transversally.

Theorem. Let us consider the equation

$$a(x,y)u_x(x,y)+b(x,y)u_y(x,y)+\lambda(x,y)u(x,y)=f(x,y),$$

where C^1-functions a, b, λ, f are defined on a domain $\Omega \subset \mathbb{R}^2$, with a side condition $u = u_0(s)$, $u_0 \in C^1$, imposed on a regular curve

$$\gamma\ :\quad x=x_0(s),\ y=y_0(s),\quad s\in I.$$

If the condition

$$x_0'(s)\,b(x_0(s),y_0(s))-y_0'(s)\,a(x_0(s),y_0(s))\neq 0\quad \forall s\in I$$

holds true, then there exists a unique solution $u = u(x,y)$ of the given equation defined on a neighborhood of the curve and satisfying the side condition

$$u(x_0(s),y_0(s))=u_0(s),\quad s\in I.$$

The curve

$$\gamma\ :\quad x=x_0(s),\ y=y_0(s),\quad s\in I$$

is called regular if $x_0, y_0 \in C^1$ and $(x_0'(s), y_0'(s)) \neq (0,0)$ for any $s \in I$.