Classification of partial differential equations

In the sequel we use the notation

$$\mathbf{x} = (x, y) \quad \text{and} \quad \mathbf{x} = (x, y, z)$$

for coordinates of 2- and 3-dimensional space variables, respectively.

The general partial differential equation in 3-dimensional case involving space and time variables:

$$F(x, y, z, t, u, u_x, u_y, u_z, u_t, u_{xx}, u_{xy}, u_{xz}, u_{xt}, ...) = 0$$

If time t is one of the independent variables of the searched-for function, then the equation is called an *evolution equation.*

If the equation contains only spatial independent variables, then the equation is called a *stationary equation.*

The highest order of the derivative of the unknown function in the equation determines the *order of the equation.*

An *operator* is a mapping that maps a function to another function.

An operator L is called *linear* if

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$$

where α, β are constants.

A differential equation is called *linear* if it has the form

$$L(u) = f$$

where L is a linear operator and f is a given function.

The linear differential equation is called
homogeneous if $f \equiv 0$
nonhomogeneous if $f \not\equiv 0$.

Basic types of partial differential equations of the second order of 2 variables:

equation of *hyperbolic type* $u_{tt} - u_{xx} = f$

equation of *parabolic type* $u_t - u_{xx} = f$

equation of *elliptic type* $u_{xx} + u_{yy} = f$

A general linear second order partial differential equation of 2 variables:

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = f$$

Matrix of coefficients of higher order terms:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

1. In case $\det A > 0$, the equation is of *elliptic* type and it is reducible to the form

$$u_{xx} + u_{yy} \ + \ b_1u_x + b_2u_y + b_0u = f$$

2. In case $\det A < 0$, the equation is of *hyperbolic* type and it is reducible to the form

$$u_{xx} - u_{yy} \ + \ b_1u_x + b_2u_y + b_0u = f$$

3. In case $\det A = 0$, the equation is of *parabolic* type and it is reducible to the form

$$u_{xx} + b_1u_x + b_2u_y + b_0u = f \quad \text{or} \quad u_{yy} + b_1u_x + b_2u_y + b_0u = f$$

These forms of the equation are called the *canonical forms.*

For simplicity, let us consider the normalized equation (divided by a_{11}):

$$u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1 u_x + a_2 u_y + a_0 u = f$$

The change of variables that reduces this equation to the canonical form:

$$x = \xi \,, \quad y = a_{12}\xi + b\eta$$

where $b = \sqrt{a_{22} - a_{12}^2}$ in the elliptic case

$b = \sqrt{a_{12}^2 - a_{22}}$ in the hyperbolic case

$b \neq 0$ - arbitrary, in the parabolic case

A general linear second order partial differential equation of $n \geq 3$ variables:

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + a_0 u = f$$

Matrix of coefficients of higher order terms:

$$A = (a_{ij})_{i,j=1,\ldots,n}$$

It is symmetric, i.e. $a_{ij} = a_{ji}$.

1. The equation is of *elliptic type*, if the eigenvalues of A are all positive or all negative. Then it is reducible to the form

   $$u_{x_1x_1} + u_{x_2x_2} + \ldots + u_{x_nx_n} \ + \text{ lower order terms} = f$$

2. The equation is of *parabolic type*, if A has exactly one zero eigenvalue and all the other eigenvalues have the same sign. Then it is reducible to the form

   $$u_{x_1x_1} + u_{x_2x_2} + \ldots + u_{x_{n-1}x_{n-1}} \ + \text{ lower order terms} = f$$

3. The equation is of *hyperbolic type*, if A has only one negative eigenvalue and all the others are positive, or A has only one positive eigenvalue and all the others are negative. Then it is reducible to the form

   $$u_{x_1x_1} - u_{x_2x_2} - \ldots - u_{x_nx_n} \ + \text{ lower order terms} = f$$

4. the equation is of *ultrahyperbolic type*, if A has more than one positive eigenvalue and more than one negative eigenvalue, and no zero eigenvalues. Then it is reducible to the form

   $$u_{x_1x_1} + u_{x_2x_2} + \ldots + u_{x_kx_k} - u_{x_{k+1}x_{k+1}} - \ldots - u_{x_nx_n} \ +$$

   $$+ \text{lower order terms} = f$$

   where $1 < k < n$.

Solutions of partial differential equations. Boundary and initial conditions.

A function u is called a *solution* of a partial differential equation if, when substituted (together with its partial derivatives) into the equation, the latter becomes an identity.

If k is the order of the given partial differential equation, then by its *classical solution* we understand a function of the space C^k satisfying the equation at each point.

By the *general solution* of a partial differential equation we understand a set of all solutions of the given equation. Very often, the general solution can be described by a formula including arbitrary functions or constants and their particular choice leads to one *particular solution* of the given equation.

In order to extract a *unique* particular solution, the partial differential equation is supplemented by additional conditions.

These conditions may be given at boundaries of physical domains (*boundary conditions*) or at initial moment of the time (*initial conditions*).

Boundary value problems for stationary equations

Consider a stationary equation in $\Omega \subset \mathbb{R}^N$, $N \in \{2; 3\}$.

Types of boundary conditions

1. *Dirichlet boundary condition or first kind boundary condition*
$$u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega$$
2. *Neumann boundary condition or second kind boundary condition*
$$\frac{\partial}{\partial \mathbf{n}} u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega$$
where $\mathbf{n}$ is the outer normal vector of $\partial\Omega$
3. *Robin boundary condition or third kind boundary condition*
$$A\frac{\partial}{\partial \mathbf{n}} u(\mathbf{x}) + Bu(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega$$
where A and B are constants, $A^2 + B^2 \neq 0$.

A boundary condition is called *homogeneous* if $g \equiv 0$

A boundary condition is called *nonhomogeneous* if $g \not\equiv 0$

Boundary value problems for *Poisson equation* $(N \in \{2; 3\})$.

Dirichlet problem

$$\Delta u(\mathbf{x}) = f(\mathbf{x}),\ \mathbf{x} \in \Omega$$
$$u(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \partial\Omega$$

Neumann problem

$$\Delta u(\mathbf{x}) = f(\mathbf{x}),\ \mathbf{x} \in \Omega$$
$$\frac{\partial}{\partial \mathbf{n}} u(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \partial\Omega$$

Robin problem

$$\Delta u(\mathbf{x}) = f(\mathbf{x}),\ \mathbf{x} \in \Omega$$
$$A\frac{\partial}{\partial \mathbf{n}} u(\mathbf{x}) + Bu(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \partial\Omega$$

Problem with *boundary conditions of mixed type*.

Let $\partial\Omega = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 \cap \Gamma_2 = \emptyset$.

The problem is formulated so that on Γ_1 and Γ_2 boundary conditions of different types are given.

For example:

$$\Delta u(\mathbf{x}) = f(\mathbf{x}),\ \mathbf{x} \in \Omega$$
$$u(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \Gamma_1$$
$$\frac{\partial}{\partial \mathbf{n}} u(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \Gamma_2$$

If the domain Ω is unbounded then additional conditions at infinity, for example

$$u(\mathbf{x}) \to 0 \quad \text{as} \quad \mathbf{x} \in \Omega,\ |\mathbf{x}| \to \infty$$

are also added.

Analogous boundary value problems can be formulated in case the Poisson equation

$$\Delta u(\mathbf{x}) = f(\mathbf{x})$$

is replaced by a more general equation of elliptic type:

$$\Delta u(\mathbf{x}) - \sum_{i=1}^{N} d_i \, u_{x_i}(\mathbf{x}) - \mu \, u(\mathbf{x}) \; = \; f(\mathbf{x})$$

Boundary value problems for *Poisson equation* in case $N = 1$.

Let $\Omega = (0, l)$.

Dirichlet problem

$$u'' = f(x),\ x \in (0, l)$$
$$u(0) = g_0,\ u(l) = g_1$$

Neumann problem

$$u'' = f(x),\ x \in (0, l)$$
$$u'(0) = g_0,\ u'(l) = g_1$$

Robin problem

$$u'' = f(x),\ x \in (0, l)$$
$$\alpha_0 u'(0) + \beta_0 u(0) = g_0,\ \alpha_1 u'(l) + \beta_1 u(l) = g_1$$

An example of a problem with boundary conditions of *mixed type*:

$$u'' = f(x),\ x \in (0, l)$$
$$u'(0) = g_0,\ u(l) = g_1$$

Analogous boundary value problems can be formulated in case the Poisson equation

$$u''(x) = f(x)$$

is replaced by a more general equation:

$$u''(x) - d\,u'(x) - \mu\,u(x) \;=\; f(x)$$

Cauchy problems for evolution equations

An evolution equation is formulated for $\mathbf{x} \in \mathbb{R}^N$,

for positive time values t

and supplemented by suitable number of initial conditions at $t = 0$.

Cauchy problem for diffusion equation

$$u_t(\mathbf{x}, t) - k\Delta u(\mathbf{x}, t) = f(\mathbf{x}, t), \ \ \mathbf{x} \in \mathbb{R}^N, \ \ t > 0$$

$$u(\mathbf{x}, 0) = \varphi(\mathbf{x}), \ \ \mathbf{x} \in \mathbb{R}^N$$

Cauchy problem for wave equation:

$$u_{tt}(\mathbf{x}, t) - c^2\Delta u(\mathbf{x}, t) = f(\mathbf{x}, t), \ \ \mathbf{x} \in \mathbb{R}^N, \ \ t > 0$$

$$u(\mathbf{x}, 0) = \varphi(\mathbf{x}), \ \ \mathbf{x} \in \mathbb{R}^N$$

$$u_t(\mathbf{x}, 0) = \psi(\mathbf{x}), \ \ \mathbf{x} \in \mathbb{R}^N$$

Initial boundary value problems for evolution equations

An evolution equation is formulated for $\mathbf{x} \in \Omega \subset \mathbb{R}^N$,
for positive time values t
and supplemented by boundary conditions at $\partial\Omega$
and a suitable number of initial conditions at $t = 0$.

An example: an initial boundary value problem for diffusion equation with Neumann boundary conditions:

$$
\begin{aligned}
& u_t(\mathbf{x},t) - k\Delta u(\mathbf{x},t) \;=\; f(\mathbf{x},t), \;\; \mathbf{x} \in \Omega, \;\; t > 0 \\
& \frac{\partial}{\partial \mathbf{n}} u(\mathbf{x},t) = g(\mathbf{x},t), \; \mathbf{x} \in \partial\Omega, \;\; t > 0 \\
& u(\mathbf{x},0) = \varphi(\mathbf{x}), \;\; \mathbf{x} \in \Omega
\end{aligned}
$$

Another example: an initial boundary value problem for equation of vibrating string with homogeneous Dirichlet boundary conditions:

$$
\begin{aligned}
& u_{tt}(x,t) - c^2 u_{xx}(x,t) \; = \; f(x,t), \;\; x \in (0,l), \;\; t > 0 \\
& u(0,t) = 0, \;\; u(l,t) = 0, \;\; t > 0 \\
& u(x,0) = \varphi(x), \;\; x \in (0,l) \\
& u_t(x,0) = \psi(x), \;\; x \in (0,l)
\end{aligned}
$$

Well-posed and ill-posed problems

A problem is *well-posed* if the following three conditions are satisfied:

(i) a solution of the problem exists;

(ii) the solution of the problem is determined uniquely;

(iii) the solution of the problem is stable with respect to the given data, which means that a small change of initial or boundary conditions, right-hand side (or other problem data) causes only a small change of the solution.

A problem is *ill-posed* if at least one of these three conditions fails.