Diffusion and wave equations in higher dimensions

1. *Cauchy problem for diffusion equation*

$$u_t(\mathbf{x},t) - k\Delta u(\mathbf{x},t) = f(\mathbf{x},t), \quad \mathbf{x} \in \mathbb{R}^N\,,\ t>0$$
$$u(\mathbf{x},0) = \varphi(\mathbf{x})$$

where $N \in \{2;3\}$.

Firstly, we consider the *homogeneous Cauchy problem* in case $N = 3$:

$$\begin{aligned} & u_t(\mathbf{x}, t) - k\Delta u(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathbb{R}^3\,,\ t > 0 \\ & u(\mathbf{x}, 0) = \varphi(\mathbf{x}) \end{aligned} \tag{1}$$

Recall that the solution of the homogeneous Cauchy problem for the diffusion equation in the one-dimensional case, i.e.

$$\begin{aligned} & u_t(x, t) - k u_{xx}(x, t) = 0, \quad x \in \mathbb{R}\,,\ t > 0 \\ & u(x, 0) = \varphi(x) \end{aligned}$$

has the following formula:

$$u(x, t) = \int_{-\infty}^{\infty} G(x - \xi, t)\varphi(\xi)d\xi$$

where

$$G(x, t) = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}}$$

is the fundamental solution.

Let us consider the problem (1) and suppose that the initial condition has the form of a function with separated variables, i.e.

$$\varphi(\mathbf{x}) = \phi(x)\psi(y)\zeta(z)\,, \quad \mathbf{x} = (x, y, z).$$

Then the solution can also be expressed in the form of separated variables, i.e.

$$u(\mathbf{x}, t) = u_1(x, t)u_2(y, t)u_3(z, t)$$

where

$$u_1(x, t) = \int_{-\infty}^{\infty} G(x - \xi, t)\phi(\xi)d\xi$$

$$u_2(y, t) = \int_{-\infty}^{\infty} G(y - \eta, t)\psi(\eta)d\eta$$

$$u_3(z, t) = \int_{-\infty}^{\infty} G(z - \theta, t)\zeta(\theta)d\theta$$

Let us check it. The functions u_1, u_2 and u_3 solve the one-dimensional Cauchy problems

$$\frac{\partial}{\partial t}u_1(x,t) - k\frac{\partial^2}{\partial x^2}u_1(x,t) = 0, \quad x \in \mathbb{R}\,,\ t > 0$$
$$u_1(x,0) = \phi(x),$$

$$\frac{\partial}{\partial t}u_2(y,t) - k\frac{\partial^2}{\partial y^2}u_2(y,t) = 0, \quad y \in \mathbb{R}\,,\ t > 0$$
$$u_2(y,0) = \psi(y)$$

and

$$\frac{\partial}{\partial t}u_3(z,t) - k\frac{\partial^2}{\partial z^2}u_3(z,t) = 0, \quad y \in \mathbb{R}\,,\ t > 0$$
$$u_3(z,0) = \zeta(z),$$

respectively.

Therefore,

$$\frac{\partial}{\partial t}u - k\Delta u = \frac{\partial}{\partial t}(u_1u_2u_3) - k\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(u_1u_2u_3) =$$

$$= u_2u_3\frac{\partial}{\partial t}u_1 + u_1u_3\frac{\partial}{\partial t}u_2 + u_1u_2\frac{\partial}{\partial t}u_3 -$$

$$-ku_2u_3\frac{\partial^2}{\partial x^2}u_1 - ku_1u_3\frac{\partial^2}{\partial y^2}u_2 - ku_1u_2\frac{\partial^2}{\partial z^2}u_3 =$$

$$= u_2u_3\left(\frac{\partial}{\partial t}u_1 - \frac{\partial^2}{\partial x^2}u_1\right) + u_1u_3\left(\frac{\partial}{\partial t}u_2 - \frac{\partial^2}{\partial y^2}u_2\right) +$$

$$+u_1u_2\left(\frac{\partial}{\partial t}u_3 - \frac{\partial^2}{\partial z^2}u_2\right) = 0.$$

Moreover,

$$u(\mathbf{x}, 0) = u_1(x, 0)u_2(y, 0)u_3(z, 0) = \phi(x)\psi(y)\zeta(z) = \varphi(\mathbf{x})$$

This shows that u solves the Cauchy problem (1).

Let make the solution formula more compact:

$$
\begin{aligned}
& u(\mathbf{x}, t) = u_1(x,t)u_2(y,t)u_3(z,t) = \\
& = \int_{-\infty}^{\infty} G(x-\xi, t)\phi(\xi)d\xi \int_{-\infty}^{\infty} G(y-\eta, t)\psi(\eta)d\eta \times \\
& \times \int_{-\infty}^{\infty} G(z-\theta, t)\zeta(\theta)d\theta = \\
& = \int_{\mathbb{R}^3} G(x-\xi, t)G(y-\eta, t)G(z-\theta, t)\phi(\xi)\psi(\eta)\zeta(\theta)d\xi d\eta d\theta = \\
& = \int_{\mathbb{R}^3} G_3(\mathbf{x}-\mathbf{y}, t)\varphi(\mathbf{y})d\mathbf{y}
\end{aligned}
$$

where $\mathbf{y} = (\xi, \eta, \theta)$ and

$$
\begin{aligned}
& G_3(\mathbf{x}, t) = G(x,t)G(y,t)G(z,t) = \\
& = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}} \frac{1}{2\sqrt{\pi k t}} e^{-\frac{y^2}{4kt}} \frac{1}{2\sqrt{\pi k t}} e^{-\frac{z^2}{4kt}} = \\
& = \frac{1}{8\sqrt{(\pi k t)^3}} e^{-\frac{x^2+y^2+z^2}{4kt}} = \frac{1}{8\sqrt{(\pi k t)^3}} e^{-\frac{|\mathbf{x}|^2}{4kt}}
\end{aligned}
$$

is the *fundamental solution* of three-dimensional diffusion equation.

Summing up, we have derived the solution formula

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^3} G_3(\mathbf{x} - \mathbf{y}, t)\varphi(\mathbf{y})d\mathbf{y}$$

for the homogeneous Cauchy problem (1) in the particular case

$$\varphi(\mathbf{x}) = \phi(x)\psi(y)\zeta(z)$$

Since the equation is linear, the same formula should be valid also for any linear combination of functions of the form $\varphi(\mathbf{x}) = \phi(x)\psi(y)\zeta(z)$.

It is possible to show that any continuous and bounded function $\varphi(\mathbf{x})$ can be approximated by functions of the form

$$\varphi_n(\mathbf{x}) = \sum_{k=1}^{n} c_k \phi_k(x)\psi_k(y)\zeta_k(z)$$

This implies that, for any bounded and continuous initial condition $\varphi(\mathbf{x})$, the solution of the homogeneous Cauchy problem (1) is represented by the formula

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^3} G_3(\mathbf{x} - \mathbf{y}, t)\varphi(\mathbf{y})d\mathbf{y}$$

By means of the operator method (as in Chapter 5), it is possible to show that the solution of the nonhomogeneous Cauchy problem

$$u_t(\mathbf{x},t) - k\Delta u(\mathbf{x},t) = f(\mathbf{x},t), \quad \mathbf{x} \in \mathbb{R}^3\,,\ t > 0$$
$$u(\mathbf{x},0) = \varphi(\mathbf{x})$$

can be expressed by the formula

$$u(\mathbf{x},t) = \int_{\mathbb{R}^3} G_3(\mathbf{x}-\mathbf{y},t)\varphi(\mathbf{y})d\mathbf{y} + \int_0^t \int_{\mathbb{R}^3} G_3(\mathbf{x}-\mathbf{y},t-s)f(\mathbf{y},s)d\mathbf{y}ds$$

Similar formulas can be deduced also in the two-dimensional case $N = 2$.

2. *Boundary value problem for diffusion equation in bounded domain*

$$
\begin{aligned}
&u_t(\mathbf{x},t) - k\Delta u(\mathbf{x},t) = f(\mathbf{x},t), \quad \mathbf{x} \in \Omega\,,\ t > 0 \\
&u(\mathbf{x},t) = h_1(\mathbf{x},t), \quad \mathbf{x} \in \Gamma_1 \\
&\tfrac{\partial}{\partial \mathbf{n}} u(\mathbf{x},t) = h_2(\mathbf{x},t), \quad \mathbf{x} \in \Gamma_2 \\
&\tfrac{\partial}{\partial \mathbf{n}} u(\mathbf{x},t) + a u(\mathbf{x},t) = h_3(\mathbf{x},t), \quad \mathbf{x} \in \Gamma_3 \\
&u(\mathbf{x},0) = \varphi(\mathbf{x})
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^N$, $N \in \{2;3\}$, $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \partial\Omega$.

By means of a proper change of variables (as in Ch. 7.3.2), the problem with nonhomogeneous boundary conditions can be transformed to a problem with homogeneous boundary conditions.

Therefore, let us consider the problem with homogeneous boundary conditions

$$
\begin{aligned}
&u_t(\mathbf{x},t) - k\Delta u(\mathbf{x},t) = f(\mathbf{x},t), \quad \mathbf{x} \in \Omega\,,\ t > 0\\
&u(\mathbf{x},t) = 0, \quad \mathbf{x} \in \Gamma_1\\
&\tfrac{\partial}{\partial \mathbf{n}} u(\mathbf{x},t) = 0, \quad \mathbf{x} \in \Gamma_2\\
&\tfrac{\partial}{\partial \mathbf{n}} u(\mathbf{x},t) + a u(\mathbf{x},t) = 0, \quad \mathbf{x} \in \Gamma_3\\
&u(\mathbf{x},0) = \varphi(\mathbf{x})
\end{aligned}
$$

In case the equation is also homogeneous, i.e. $f(\mathbf{x}, t) \equiv 0$, we can use the method of *separation of variables.*

Suppose that the solution has the form

$$u(\mathbf{x}, t) = V(\mathbf{x})T(t)$$

Let us insert this solution into the equation and separate t-dependent function T and x-dependent function X:

$$V(\mathbf{x})T'(t) - k\Delta V(\mathbf{x})T(t) = 0 \;\Rightarrow$$

$$\frac{T'(t)}{kT(t)} = \frac{\Delta V(\mathbf{x})}{V(\mathbf{x})}$$

Consequently,

$$-\frac{T'(t)}{kT(t)} = -\frac{\Delta V(\mathbf{x})}{V(\mathbf{x})} \; = \; \lambda$$

where λ is a constant.

We obtain 2 equations

$$\Delta V(\mathbf{x}) + \lambda V(\mathbf{x}) = 0$$
$$T' + k\lambda T = 0$$

Let us consider the first equation with given homogeneous Dirichlet boundary conditions:

$$\begin{aligned}
&\Delta V(\mathbf{x}) + \lambda V(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \\
&V(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_1 \\
&\tfrac{\partial}{\partial \mathbf{n}} V(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_2 \\
&\tfrac{\partial}{\partial \mathbf{n}} V(\mathbf{x}) + aV(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_3
\end{aligned}$$

This is an *eigenvalue problem* for the operator $-\Delta$.

It can be shown that this problem has an infinite sequence of nonnegative eigenvalues

$$\lambda_n \to \infty \quad \text{as} \quad n \to \infty$$

and a corresponding complete system of orthogonal eigenfunctions $V_n(\mathbf{x})$.

Let us consider the second equation

$$T'(t) + k\lambda T(t) = 0$$

The general solution corresponding to $\lambda = \lambda_n$ is

$$T(t) = T_n(t) = A_n e^{-k\lambda_n t}$$

where A_n is an arbitrary constant.

We have obtained the following family of solutions of the equation $u_t - k\Delta u = 0$ that satisfy the homogeneous boundary conditions:

$$u(\mathbf{x}, t) = u_n(\mathbf{x}, t) = A_n e^{-k\lambda_n t} V_n(\mathbf{x}), \quad n = 1, 2, \ldots$$

where A_n are arbitrary constants.

Since the equation is linear, the following series

$$u(\mathbf{x}, t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda_n t} V_n(\mathbf{x})$$

is also the solution of the equation $u_t - k\Delta u = 0$.

It satisfies the homogeneous boundary conditions, too.

Setting $t = 0$, we have

$$\varphi(\mathbf{x}) = u(\mathbf{x}, 0) = \sum_{n=1}^{\infty} A_n V_n(\mathbf{x})$$

Due to the orthogonality of eigenfunctions, the coefficients A_n can be expressed as

$$A_n = \frac{\int\limits_{\Omega} \varphi(\mathbf{x}) V_n(\mathbf{x}) d\mathbf{x}}{\int\limits_{\Omega} V_n^2(\mathbf{x}) d\mathbf{x}}$$

Let us also consider the problem with nonhomogeneous equation $u_t - k\Delta u = f$.
Then the solution can be expressed as

$$u(\mathbf{x},t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda_n t}\, V_n(\mathbf{x}) + \sum_{n=1}^{\infty} \int_0^t e^{-k\lambda_n (t-s)} f_n(s) ds\, V_n(\mathbf{x})$$

where

$$A_n = \frac{\int\limits_{\Omega} \varphi(\mathbf{x}) V_n(\mathbf{x}) d\mathbf{x}}{\int\limits_{\Omega} V_n^2(\mathbf{x}) d\mathbf{x}}$$

$$f_n(t) = \frac{\int\limits_{\Omega} f(\mathbf{x},t) V_n(\mathbf{x}) d\mathbf{x}}{\int\limits_{\Omega} V_n^2(\mathbf{x}) d\mathbf{x}}$$

3. *Cauchy problem for wave equation*

$$
\begin{aligned}
& u_{tt}(\mathbf{x},t) - c^2 \Delta u(\mathbf{x},t) = f(\mathbf{x},t), \quad \mathbf{x} \in \mathbb{R}^N\,,\ t > 0 \\
& u(\mathbf{x},0) = \varphi(\mathbf{x}) \\
& u_t(\mathbf{x},0) = \psi(\mathbf{x})
\end{aligned}
$$

where $N \in \{2;3\}$.

Homogeneous Cauchy problem in case $N = 3$:

$$u_{tt}(\mathbf{x}, t) - c^2 \Delta u(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathbb{R}^3\,,\ t > 0$$
$$u(\mathbf{x}, 0) = \varphi(\mathbf{x})$$
$$u_t(\mathbf{x}, 0) = \psi(\mathbf{x})$$

The solution of this problem can be expressed by means of the *Kirchhoff's formula*

$$u(\mathbf{x}, t) = \frac{1}{4\pi c^2 t} \int\limits_{|\mathbf{x}-\mathbf{y}|=ct} \psi(\mathbf{y}) ds + \frac{\partial}{\partial t} \left(\frac{1}{4\pi c^2 t} \int\limits_{|\mathbf{x}-\mathbf{y}|=ct} \varphi(\mathbf{y}) ds \right)$$

Huygens principle.

According to Kirchhoffs formula, the solution of u at the point $(\mathbf{x}, t)$ depends only on the values of $\varphi(\mathbf{y})$ and $\psi(\mathbf{y})$ for $\mathbf{y}$ from the spherical surface $|\mathbf{x} - \mathbf{y}| = ct$, but it does not depend on the values of the initial data inside this sphere. Similarly, using the opposite point of view we conclude that the values of φ and ψ at a point $\mathbf{y} \in \mathbb{R}^3$ influence the solution of the three-dimensional wave equation only on the spherical surface $|\mathbf{x} - \mathbf{y}| = ct$.

Solution formula for nonhomogeneous Cauchy problem:

$$u(\mathbf{x},t) = \frac{1}{4\pi c^2 t} \int\limits_{|\mathbf{x}-\mathbf{y}|=ct} \psi(\mathbf{y}) ds + \frac{\partial}{\partial t} \left(\frac{1}{4\pi c^2 t} \int\limits_{|\mathbf{x}-\mathbf{y}|=ct} \varphi(\mathbf{y}) ds \right) +$$

$$+ \frac{1}{4\pi c} \int\limits_{|\mathbf{x}-\mathbf{y}|\leq ct} \frac{f(\mathbf{y}, t - \frac{1}{c}|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y}$$

4. *Boundary value problem for wave equation in bounded domain*

$$u_{tt}(\mathbf{x},t) - c^2 \Delta u(\mathbf{x},t) = f(\mathbf{x},t), \quad \mathbf{x} \in \Omega\,,\ t > 0$$
$$u(\mathbf{x},t) = h_1(\mathbf{x},t), \quad \mathbf{x} \in \Gamma_1$$
$$\frac{\partial}{\partial \mathbf{n}} u(\mathbf{x},t) = h_2(\mathbf{x},t), \quad \mathbf{x} \in \Gamma_2$$
$$\frac{\partial}{\partial \mathbf{n}} u(\mathbf{x},t) + a u(\mathbf{x},t) = h_3(\mathbf{x},t), \quad \mathbf{x} \in \Gamma_3$$
$$u(\mathbf{x},0) = \varphi(\mathbf{x})$$
$$u_t(\mathbf{x},0) = \psi(\mathbf{x})$$

where $\Omega \subset \mathbb{R}^N$, $N \in \{2;3\}$, $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \partial\Omega$.

By means of a change of variables, the problem with non-homogeneous boundary conditions can be transformed to a problem with homogeneous boundary conditions.

$$u_{tt}(\mathbf{x},t) - c^2 \Delta u(\mathbf{x},t) = f(\mathbf{x},t), \quad \mathbf{x} \in \Omega\,,\; t > 0$$

$$u(\mathbf{x},t) = 0, \quad \mathbf{x} \in \Gamma_1$$

$$\frac{\partial}{\partial \mathbf{n}} u(\mathbf{x},t) = 0, \quad \mathbf{x} \in \Gamma_2$$

$$\frac{\partial}{\partial \mathbf{n}} u(\mathbf{x},t) + a u(\mathbf{x},t) = 0, \quad \mathbf{x} \in \Gamma_3$$

$$u(\mathbf{x},0) = \varphi(\mathbf{x})$$

$$u_t(\mathbf{x},0) = \psi(\mathbf{x})$$

In case the equation is also homogeneous, i.e. $f(\mathbf{x}, t) \equiv 0$, we can use the method of *separation of variables*.

Suppose that the solution has the form

$$u(\mathbf{x}, t) = V(\mathbf{x})T(t)$$

Let us insert this solution into the equation and separate t-dependent function T and x-dependent function X:

$$V(\mathbf{x})T''(t) - c^2 \Delta V(\mathbf{x})T(t) = 0 \;\Rightarrow$$

$$\frac{T''(t)}{c^2 T(t)} = \frac{\Delta V(\mathbf{x})}{V(\mathbf{x})}$$

Consequently,

$$-\frac{T''(t)}{c^2 T(t)} = -\frac{\Delta V(\mathbf{x})}{V(\mathbf{x})} \; = \; \lambda$$

where λ is a constant.

We obtain 2 equations

$$
\begin{aligned}
&\Delta V(\mathbf{x}) + \lambda V(\mathbf{x}) = 0 \\
&T'' + c^2 \lambda T = 0
\end{aligned}
$$

Let us consider the first equation with given homogeneous Dirichlet boundary conditions:

$$
\begin{aligned}
&\Delta V(\mathbf{x}) + \lambda V(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \\
&V(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_1 \\
&\tfrac{\partial}{\partial \mathbf{n}} V(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_2 \\
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\end{aligned}
$$

Such an *eigenvalue problem* for the operator $-\Delta$ we already considered.

It has an infinite sequence of nonnegative eigenvalues

$$
\lambda_n \to \infty \quad \text{as} \quad n \to \infty
$$

and a corresponding complete system of orthogonal eigenfunctions $V_n(\mathbf{x})$.

Let us consider the second equation

$$T''(t) + c^2\lambda T(t) = 0$$

The general solution corresponding to $\lambda = \lambda_n$ is

$$T(t) = T_n(t) = A_n \cos(c\sqrt{\lambda_n} t) + B_n \sin(c\sqrt{\lambda_n} t)$$

where A_n, B_n are arbitrary constants.

We have obtained the following family of solutions of the equation $u_{tt} - c^2\Delta u = 0$ that satisfy the homogeneous boundary conditions:

$$u(\mathbf{x}, t) = u_n(\mathbf{x}, t) = [A_n \cos(c\sqrt{\lambda_n} t) + B_n \sin(c\sqrt{\lambda_n} t)]V_n(\mathbf{x}), \quad n = 1, 2, \ldots$$

where A_n, B_n are arbitrary constants.

Since the equation is linear, the following series

$$u(\mathbf{x},t) = \sum_{n=1}^{\infty} [A_n \cos(c\sqrt{\lambda_n} t) + B_n \sin(c\sqrt{\lambda_n} t)] V_n(\mathbf{x})$$

is also the solution of the equation $u_{tt} - c^2 \Delta u = 0$.

It satisfies the homogeneous boundary conditions, too.

Using the boundary conditions and orthogonality of eigenfunctions, we can deduce the formulas for coefficients A_n and B_n:

$$A_n = \frac{\int\limits_{\Omega} \varphi(\mathbf{x}) V_n(\mathbf{x}) d\mathbf{x}}{\int\limits_{\Omega} V_n^2(\mathbf{x}) d\mathbf{x}}$$

$$B_n = \frac{\int\limits_{\Omega} \psi(\mathbf{x}) V_n(\mathbf{x}) d\mathbf{x}}{c\sqrt{\lambda_n} \int\limits_{\Omega} V_n^2(\mathbf{x}) d\mathbf{x}}$$