Laplace and Poisson equations in higher dimensions. Boundary value problems.

Poisson equation

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^N \quad N \in \{2; 3\}$$

Laplace equation

$$\Delta u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^N \quad N \in \{2; 3\}$$

Green's first identity.

Let us consider the divergence theorem applied to a vector field $\mathbf{V}(\mathbf{x})$:

$$\int_{\Omega} \operatorname{div} \mathbf{V}(\mathbf{x}) d\mathbf{x} \;=\; \int_{\partial\Omega} \mathbf{V}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) d\mathbf{s}$$

Set here

$$\mathbf{V}(x) = v(x) \nabla u(x)$$

where v and u are some scalar functions.
Then

$$\operatorname{div} \mathbf{V} = \nabla v \cdot \nabla u + v \Delta u \,, \quad \mathbf{V} \cdot \mathbf{n} = v \, \nabla u \cdot \mathbf{n} = v \frac{\partial u}{\partial \mathbf{n}}$$

We obtain the following formula:

$$\int_{\partial\Omega} v(\mathbf{x}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) d\mathbf{s} = \int_{\Omega} \nabla v(\mathbf{x}) \cdot \nabla u(\mathbf{x}) d\mathbf{x} + \int_{\Omega} v(\mathbf{x}) \Delta u(\mathbf{x}) d\mathbf{x}$$

It is called the Green's first identity

Energy method. Uniqueness of solutions of boundary value problems.

Consider the Poisson equation

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

Replacing v by u and Δu by f in Green's first identity, we have

$$\int_{\Omega} |\nabla u(\mathbf{x})|^2 d\mathbf{x} = \int_{\partial\Omega} u(\mathbf{x}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) d\mathbf{s} - \int_{\Omega} f(\mathbf{x}) u(\mathbf{x}) d\mathbf{x}$$

This is an energy relation - the term $\int_{\Omega} |\nabla u(\mathbf{x})|^2 d\mathbf{x}$ in the left- hand side is an equivalent of total potential energy of the system.

Dirichlet problem

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \;\; \mathbf{x} \in \Omega, \qquad u(\mathbf{x}) = g(\mathbf{x}), \;\; \mathbf{x} \in \partial\Omega$$

Suppose that this problem has two solutions u_1 and u_2. Then

$$\Delta u_1(\mathbf{x}) = f(\mathbf{x}), \;\; \mathbf{x} \in \Omega, \qquad u_1(\mathbf{x}) = g(\mathbf{x}), \;\; \mathbf{x} \in \partial\Omega$$
$$\Delta u_2(\mathbf{x}) = f(\mathbf{x}), \;\; \mathbf{x} \in \Omega, \qquad u_2(\mathbf{x}) = g(\mathbf{x}), \;\; \mathbf{x} \in \partial\Omega$$

Subtracting we obtain the homogeneous problem for the difference $u = u_1 - u_2$:

$$\Delta u(\mathbf{x}) = 0, \;\; \mathbf{x} \in \Omega, \qquad u(\mathbf{x}) = 0, \;\; \mathbf{x} \in \partial\Omega$$

The energy relation implies

$$\int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x} = 0.$$

If an intergal of a nonegative function is zero, then this function is identically zero.

Therefore,

$$|\nabla u(\mathbf{x})| \equiv 0$$

This implies $\nabla u(\mathbf{x}) \equiv 0$. Consequently,

$$u(\mathbf{x}) \equiv C, \quad \text{where } C \text{ is a constant}$$

Since $u(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$, we have $C = 0$ and $u(\mathbf{x}) \equiv 0$.

This proves that $u_1(\mathbf{x}) \equiv u_2(\mathbf{x})$. The solution of Dirichlet problem is unique.

Neumann problem

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \;\; \mathbf{x} \in \Omega, \qquad \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = h(\mathbf{x}), \;\; \mathbf{x} \in \partial\Omega$$

Analogously, as in the case of Dirichlet problem, we can show that the difference

$$u = u_1 - u_2$$

of two solutions u_1 and u_2 of this problem satisfies

$$u(\mathbf{x}) \equiv C, \;\; \text{where } C \text{ is a constant}$$

And this is all.

Solution of Neumann problem for Poisson equation is determined with a precision of the added constant.

The source density f and given flux h at the boundary must satisfy a certain consistency condition.

Setting $v = 1$ in Green's first identity, we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) d\mathbf{s} = \int_{\Omega} \Delta u(\mathbf{x}) d\mathbf{x}$$

Substituting $\frac{\partial u}{\partial \mathbf{n}}$ by h and Δu by f, we obtain

$$\int_{\partial\Omega} h(\mathbf{x}) d\mathbf{s} = \int_{\Omega} f(\mathbf{x}) d\mathbf{x}$$

Provided the amount of substance remains unchanged in the domain Ω, the amount of substance gained from sources within a unit time must equal the amount of substance flowing out from the boundary within a unit time.

The deduced consistency relation for f and h is also a necessary condition for the existence of a solution of the Neumann problem.

Solution formulas of boundary value problems.

Green's second identity.

Take the Green's first identity

$$\int_{\partial\Omega} v(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})d\mathbf{s} = \int_{\Omega} \nabla v(\mathbf{x})\cdot\nabla u(\mathbf{x})d\mathbf{x} + \int_{\Omega} v(\mathbf{x})\Delta u(\mathbf{x})d\mathbf{x}$$

Interchange u and v:

$$\int_{\partial\Omega} u(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x})d\mathbf{s} = \int_{\Omega} \nabla u(\mathbf{x})\cdot\nabla v(\mathbf{x})d\mathbf{x} + \int_{\Omega} u(\mathbf{x})\Delta v(\mathbf{x})d\mathbf{x}$$

Subtract from the lower formula the upper one:

$$\int_{\partial\Omega} \left[u(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x}) - v(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})\right] d\mathbf{s} = \int_{\Omega} \left[u(\mathbf{x})\Delta v(\mathbf{x}) - v(\mathbf{x})\Delta u(\mathbf{x})\right] d\mathbf{x}$$

Thus,

$$\int_{\Omega} \left[u(\mathbf{x})\Delta v(\mathbf{x}) - v(\mathbf{x})\Delta u(\mathbf{x})\right] d\mathbf{x} = \int_{\partial\Omega} \left[u(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x}) - v(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})\right] d\mathbf{s}$$

This relation is called the Green's second identity.

Dirac delta function in multiple dimensions and fundamental solution of Laplace equation.

The dirac delta function $\delta(\mathbf{x})$ in cases $N \in \{2;3\}$ is such a function that the equality

$$\int_{\mathbb{R}^N} \delta(\mathbf{y}) f(\mathbf{y}) d\mathbf{y} = f(\mathbf{0})$$

is valid for any continuous function f.

Intuitively,

$$\delta(\mathbf{x}) \; = \; \begin{cases} 0 & \text{if } \; \mathbf{x} \neq \mathbf{0} \\ \infty & \text{if } \; \mathbf{x} = \mathbf{0} \end{cases}$$

It holds

$$\int_{\mathbb{R}^N} \delta(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y} \; = \; f(\mathbf{x})$$

for any continuous function f.

Fundamental solution of the Laplace equation is a function Φ that satisfies the equation

$$\Delta\Phi(\mathbf{x}) = \delta(\mathbf{x})\,, \quad \mathbf{x} \in \mathbb{R}^N.$$

It is possible to prove that

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|} \qquad \text{in case } N = 3$$

$$\Phi(\mathbf{x}) = \frac{1}{2\pi}\ln|\mathbf{x}| \qquad \text{in case } N = 2$$

are fundamental solutions of the Laplace equation.

Preliminary solution formula.

Let us be limited to the case $N = 3$ (the case $N = 2$ is similar).

Suppose that u is a solution of the Poisson equation

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^3$$

Let us plug $u(\mathbf{y})$ and $v(\mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y})$ into the Green's second identity

$$\int_\Omega [u(\mathbf{y})\Delta v(\mathbf{y}) - v(\mathbf{y})\Delta u(\mathbf{y})]\, d\mathbf{y} = \int_{\partial\Omega} \left[u(\mathbf{y})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{y}) - v(\mathbf{y})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\right] d\mathbf{s}$$

We obtain

$$\int_\Omega [u(\mathbf{y})\Delta\Phi(\mathbf{x} - \mathbf{y}) - \Phi(\mathbf{x} - \mathbf{y})\Delta u(\mathbf{y})]\, d\mathbf{y} =$$
$$= \int_{\partial\Omega} \left[u(\mathbf{y})\frac{\partial}{\partial \mathbf{n}}\Phi(\mathbf{x} - \mathbf{y}) - \Phi(\mathbf{x} - \mathbf{y})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\right] d\mathbf{s}$$

Since

$$\Delta\Phi(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) \quad \text{and} \quad \Delta u(\mathbf{y}) = f(\mathbf{y})$$

we have

$$\int_\Omega u(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})d\mathbf{y} - \int_\Omega \Phi(\mathbf{x}-\mathbf{y})f(\mathbf{y})d\mathbf{y} =$$
$$= \int_{\partial\Omega} \left[u(\mathbf{y})\frac{\partial}{\partial \mathbf{n}}\Phi(\mathbf{x}-\mathbf{y}) - \Phi(\mathbf{x}-\mathbf{y})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) \right] d\mathbf{s}$$

Since $\int_\Omega u(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})d\mathbf{y} = u(\mathbf{x})$ we get

$$u(\mathbf{x}) - \int_\Omega \Phi(\mathbf{x}-\mathbf{y})f(\mathbf{y})d\mathbf{y} =$$
$$= \int_{\partial\Omega} \left[u(\mathbf{y})\frac{\partial}{\partial \mathbf{n}}\Phi(\mathbf{x}-\mathbf{y}) - \Phi(\mathbf{x}-\mathbf{y})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) \right] d\mathbf{s}$$

and

$$u(\mathbf{x}) = \int_\Omega \Phi(\mathbf{x}-\mathbf{y})f(\mathbf{y})d\mathbf{y} +$$
$$+ \int_{\partial\Omega} \left[u(\mathbf{y})\frac{\partial}{\partial \mathbf{n}}\Phi(\mathbf{x}-\mathbf{y}) - \Phi(\mathbf{x}-\mathbf{y})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) \right] d\mathbf{s}$$

The fundamental solution is $\Phi(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}$.

Consequently,

$$u(\mathbf{x}) = -\frac{1}{4\pi}\int_{\Omega}\frac{1}{|\mathbf{x}-\mathbf{y}|}f(\mathbf{y})d\mathbf{y} +$$

$$+\frac{1}{4\pi}\int_{\partial\Omega}\left[\frac{1}{|\mathbf{x}-\mathbf{y}|}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) - \frac{\partial}{\partial \mathbf{n}}\frac{1}{|\mathbf{x}-\mathbf{y}|}u(\mathbf{y})\right]d\mathbf{s}$$

This is a preliminary solution formula for a boundary value problem for Poisson equation.

It is applicable in case we are given:

1. density of sources f;

2. both the Dirichlet and Neumann boundary conditions $u(\mathbf{y})$ and $\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})$ on the boundary $\partial\Omega$.

Usually only one boundary condition is given.

Instead of $\Phi(\mathbf{x}-\mathbf{y}) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$
we should use a function $G(\mathbf{x},\mathbf{y})$ such that

$$\text{either} \quad G(\mathbf{x},\mathbf{y}) = 0 \quad \text{or} \quad \frac{\partial}{\partial \mathbf{n}} G(\mathbf{x},\mathbf{y}) = 0 \quad \text{on} \;\; \partial\Omega$$

The *Green's function* of the Laplace operator corresponding to the Dirichlet boundary condition is a function $G(\mathbf{x},\mathbf{y})$ that has the form

$$G(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} + H(\mathbf{x},\mathbf{y})$$

where H is a harmonic function, i.e.

$$\Delta_y H(\mathbf{x},\mathbf{y}) = \left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2} \right) H(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{y} \in \Omega,$$

and satisfies the boundary condition

$$G(\mathbf{x},\mathbf{y}) = 0, \qquad \mathbf{y} \in \partial\Omega.$$

Boundary condition for H:

$$H(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}, \quad \mathbf{y} \in \partial\Omega$$

Suppose that $G(\mathbf{x}, \mathbf{y})$ is the Green's function of the Laplace operator corresponding to the Dirichlet boundary condition.

Let us plug $u(\mathbf{y})$ and $v(\mathbf{y}) = G(\mathbf{x}, \mathbf{y})$ into the Green's second identity

$$\int_{\Omega} [u(\mathbf{y})\Delta v(\mathbf{y}) - v(\mathbf{y})\Delta u(\mathbf{y})]\, d\mathbf{y} = \int_{\partial\Omega} \left[u(\mathbf{y})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{y}) - v(\mathbf{y})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\right] d\mathbf{s}$$

By means of similar computations as before, we reach the following formula for u:

$$u(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y} + \int_{\partial\Omega} \frac{\partial}{\partial \mathbf{n}} G(\mathbf{x}, \mathbf{y})\, u(\mathbf{y}) d\mathbf{s}$$

Consequently, the formula of a solution of the Poisson equation with Dirichlet boundary condition

$$u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega$$

is

$$u(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y} + \int_{\partial\Omega} \frac{\partial}{\partial \mathbf{n}} G(\mathbf{x}, \mathbf{y})\, g(\mathbf{y}) d\mathbf{s}$$

The formula of the fundamental solution is simple:

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}.$$

But formulas of Green's functions are quite complicated.

Simple Green's functions occur only in cases of special geometry of Ω, i.e. a half-space, a ball.

The method of reflection can be used to derive formulas for Green's functions (see the textbook).