Some general mathematics

$\Omega \subset \mathbb{R}^N, \; N = 1, 2, 3$ - a domain

(may be bounded or unbounded)

$\mathbf{x} = (x_1, \ldots, x_N)$ - a point in Ω

$\partial\Omega$ - boundary of the domain Ω

In case $N = 1$, a bounded domain Ω is an interval

$\Omega = (0, l)$ and $\partial\Omega = \{0; l\}$.

Some notation

Notation of partial derivatives

$$u_{x_i}(\mathbf{x}) = \partial_{x_i} u(\mathbf{x}) = \frac{\partial}{\partial x_i} u(\mathbf{x})$$

Notation of mutiple integrals

$$\int_\Omega f(\mathbf{x}) d\mathbf{x}$$

In case $N = 3$

$$\int_\Omega f(\mathbf{x}) d\mathbf{x} = \iiint\limits_\Omega f(x_2, x_2, x_3) dx_1 dx_2 dx_3$$

In case $N = 2$

$$\int_\Omega f(\mathbf{x}) d\mathbf{x} = \iint\limits_\Omega f(x_2, x_2) dx_1 dx_2$$

In case $N = 1$ and $\Omega = (0, l)$

$$\int_{\Omega} f(x)dx = \int_0^l f(x)dx$$

A notation of a generalized surface integral

$$\int_{\partial\Omega} f(\mathbf{x})d\mathbf{s}$$

In case $N = 3$ it is a surface integral.

In case $N = 2$ it is a line integral.

Spaces of functions

$C(\Omega)$ - space of functions that are continuous in the domain Ω

$C^1(\Omega)$ - space of functions that are continuous with their first order (partial) derivatives in the domain Ω

$C^k(\Omega)$ - space of functions that are continuous with their (partial) derivatives up to the order k in the domain Ω.

In case $\Omega = \mathbb{R}^n$, we use the abbreviated notation $C(\mathbb{R}^N) = C$, $C^k(\mathbb{R}^N) = C^k$, too.

$L_2(\Omega)$ - space of functions $u(\mathbf{x})$ such that the integral

$$\int_\Omega u^2(\mathbf{x})d\mathbf{x}$$

exists and is finite.

$L_2(\Omega)$ is a Hilbert space with inner product

$$\langle u, v\rangle = \int_\Omega u(\mathbf{x})v(\mathbf{x})d\mathbf{x}$$

Moving from integral relations to pointwise relations

Let $f \in C$ and

$$\int_{\Omega} f(\mathbf{x}) d\mathbf{x} = 0$$

for any domain Ω. Then

$$f(\mathbf{x}) = 0$$

for any $\mathbf{x}$.

Divergence and divergence theorem.

Let $\mathbf{v}(\mathbf{x}) = (v_1(\mathbf{x}), \ldots, v_N(\mathbf{x}))$ be a vector depending on $\mathbf{x}$.

The divergence of $\mathbf{v}$ is given by

$$\operatorname{div} \mathbf{v} \; = \; \nabla \cdot \mathbf{v} \; = \; \sum_{i=1}^{N} \frac{\partial v_i}{\partial x_i}$$

Let $\mathbf{n}(\mathbf{x})$ be the outer normal vector of the surface $\partial\Omega$ at the point $\mathbf{x} \in \partial\Omega$.

The divergence theorem:

$$\int_{\Omega} \operatorname{div} \mathbf{v}(\mathbf{x}) d\mathbf{x} \; = \; \int_{\partial\Omega} \mathbf{v}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) d\mathbf{s}$$

In case $N = 1$ this theorem takes the form

$$\int_0^l v'(x) dx \; = \; v(l) - v(0)$$

The Laplace operator

$$\Delta = \operatorname{div}\operatorname{grad} = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$$

In the one-dimensional case

$$\Delta = \frac{d^2}{dx^2}$$

A conservation law

Let us consider the three-dimensional case, i.e. $N = 3$.
Let $\Omega \subset \mathbb{R}^3$ be a physical domain.

$u(\mathbf{x}, t)$ - concentration of a physical substance at a point $\mathbf{x} \in \Omega$ and a time moment $t \geq 0$.

Let $\Omega_B \subset \Omega$. We call it a "balance domain".

$\int_{\Omega_B} u(\mathbf{x}, t) d\mathbf{x}$ is the amount of the substance in the domain Ω_B at the time moment t.

Change of the amount of the substance in the domain Ω_B in the time interval $[t_1, t_2]$:

$$\int_{\Omega_B} u(\mathbf{x}, t_2) d\mathbf{x} - \int_{\Omega_B} u(\mathbf{x}, t_1) d\mathbf{x}$$

Balance of the substance (conservation law):

$$\int_{\Omega_B} u(\mathbf{x}, t_2) d\mathbf{x} - \int_{\Omega_B} u(\mathbf{x}, t_1) d\mathbf{x} =$$

$$= [\text{inflow of substance from the boundary } \partial\Omega_B] +$$

$$+ [\text{gain from sources inside } \Omega_B]$$

Equivalent form of the conservation law:

$$\int_{\Omega_B} u(\mathbf{x}, t_2) d\mathbf{x} - \int_{\Omega_B} u(\mathbf{x}, t_1) d\mathbf{x} =$$

$$= - [\text{outflow of substance from the boundary } \partial\Omega_B] +$$

$$+ [\text{gain from sources inside } \Omega_B]$$

Sources

Let V be a small part of the domain surrounding a point $\mathbf{x}$.

Let V stand for the volume of V, too.

By $F_{V,\delta}(x,t)$ we denote an amount of the substance coming from sources located in V in a time interval $[t, t+\delta]$.

Let us consider the process when $V \to 0$, i.e. the set V vanishes to the point $\mathbf{x}$ and $\delta \to 0$.

The limit

$$f(\mathbf{x}, t) = \lim_{\substack{V\to 0 \\ \delta\to 0}} \frac{F_{V,\delta}}{V\delta}$$

is called the source function (in other words: it is a density of rate of sources).

The gain of substance from sources inside the domain Ω_B in the time interval $[t_1, t_2]$ is

$$\int_{t_1}^{t_2} \int_{\Omega_B} f(\mathbf{x}, t) d\mathbf{x} dt$$

Flow and flux

Let $\mathbf{e}_i$ be a unit vector pointing to the direction of the axis x_i.

Let S be a small piece of a plain whose normal vector is $\mathbf{e}_i$ and contains a point $\mathbf{x}$.

Let S stand for the area of S, too.

We denote by $\Phi_{S,\delta}$ the amount of substance flowing through S in a time interval $[t, t+\delta]$.

Let us consider the process when $S \to 0$, i.e. the set S vanishes to the point $\mathbf{x}$ and $\delta \to 0$.

The limit

$$\phi_i(\mathbf{x}, t) = \lim_{\substack{S \to 0 \\ \delta \to 0}} \frac{\Phi_{S,\delta}}{S\delta}$$

is called the flux of the substance in the direction $\mathbf{e}_i$.

The vector

$$\phi(\mathbf{x}, t) = (\phi_1(\mathbf{x}, t), \phi_2(\mathbf{x}, t), \phi_3(\mathbf{x}, t))$$

is called the flux vector.

Let $\mathbf{v}$ be a unit vector ($|\mathbf{v}| = 1$).

The inner product

$$\phi \cdot \mathbf{v}$$

represents the flux in the direction of the vector $\mathbf{v}$.

Let $\mathbf{n}(\mathbf{x})$ be the outer normal vector of the surface $\partial\Omega_B$ at the point $\mathbf{x} \in \partial\Omega_B$.

The outflow of substance through the surface $\partial\Omega_B$ in the time interval $[t_1, t_2]$ is given by the formula:

$$\int_{t_1}^{t_2} \int_{\partial\Omega_B} \phi(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) d\mathbf{s} dt$$

The conservation law in the *integral form*:

$$\int_{\Omega_B} u(\mathbf{x}, t_2) d\mathbf{x} - \int_{\Omega_B} u(\mathbf{x}, t_1) d\mathbf{x} =$$

$$= - \int_{t_1}^{t_2} \int_{\partial\Omega_B} \phi(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) d\mathbf{s} dt \ +$$

$$+ \int_{t_1}^{t_2} \int_{\Omega_B} f(\mathbf{x}, t) d\mathbf{x} dt$$

Using the divergence theorem and the formula

$$u(\mathbf{x}, t_2) - u(\mathbf{x}, t_1) = \int_{t_1}^{t_2} u_t(\mathbf{x}, t) dt$$

we obtain

$$\int_{t_1}^{t_2} \int_{\Omega_B} \left[u_t(\mathbf{x}, t) + \operatorname{div} \phi(\mathbf{x}, t) - f(\mathbf{x}, t) \right] d\mathbf{x} dt = 0$$

This implies the conservation law in the *differential form*:

$$u_t(\mathbf{x}, t) + \operatorname{div} \phi(\mathbf{x}, t) \ = \ f(\mathbf{x}, t)$$

Two- and onedimensional cases

In *two-dimensional case* the flux ϕ is defined in an analogous manner via flow through $\partial\Omega_B$ that is a line.

The conservation law has the same form as in the 3-dimensional case:

$$u_t(\mathbf{x}, t) + \operatorname{div} \phi(\mathbf{x}, t) \ = \ f(\mathbf{x}, t)$$

In *one-dimensional case*, $\Omega = (0, l)$, $\partial\Omega = \{0; l\}$.

Let the balance domain be $\Omega_B = (a, b) \subset (0, l)$

and consider the process in a time interval $[t_1, t_2]$.

As in the 3-dimensional case, the basic conservation law reads as

$$\int_a^b u(x, t_2)dx - \int_a^b u(x, t_1)dx =$$

$$= -\ [\text{outflow of substance from the boundary } \{a; b\}] \ +$$

$$+\ [\text{gain from sources inside } (a, b)]$$

Gain from sources inside (a, b) in the time interval $[t_1, t_2]$ is

$$\int_{t_1}^{t_2} \int_a^b f(x,t)dxdt$$

where f is a source function.

Let Φ_δ denote an amount of the substance moving through a point x to the positive direction in time interval $[t, t+\delta]$.

Flux is the flow rate and is defined by the limit

$$\phi(x,t) = \lim_{\delta \to 0} \frac{\Phi_\delta}{\delta}$$

Outflow of substance from the boundary $\{a; b\}$ in the time interval $[t_1, t_2]$ is equal to

$$\int_{t_1}^{t_2} \phi(b,t)dt - \int_{t_1}^{t_2} \phi(a,t)dt$$

The conservation law in the *integral form*:

$$\int_a^b u(x,t_2)dx - \int_a^b u(x,t_1)dx =$$

$$= - \int_{t_1}^{t_2} \phi(b,t)dt + \int_{t_1}^{t_2} \phi(a,t)dt \ +$$

$$+ \int_{t_1}^{t_2} \int_a^b f(x,t)dxdt$$

Using Newton-Leibnitz formula, we obtain

$$\int_{t_1}^{t_2} \int_a^b \left[u_t(x,t) + \phi_x(x,t) - f(x,t)\right] dxdt = 0$$

This implies the conservation law in the *differential form*:

$$u_t(x,t) + \phi_x(x,t) \ = \ f(x,t)$$

Constitutive laws and basic equations of mathematical physics

Transport equation

Drift of a substance in a tube. Then

$$\phi = cu$$

where $c > 0$ is a constant

From the conservation law

$$u_t(x,t) + \phi_x(x,t) \; = \; f(x,t)$$

we deduce the transport equation

$$u_t(x,t) + c\, u_x(x,t) \; = \; f(x,t)$$

If sources are absent then

$$u_t(x,t) + c\, u_x(x,t) \; = \; 0$$

Transport with decay:

$$f = -\lambda u$$

where $\lambda > 0$ is a constant.

Then the equation is

$$u_t(x,t) + c\,u_x(x,t) + \lambda u(x,t) = 0$$

One-dimensional diffusion

Constitutive law (Fick's law):

$$\phi = -ku_x$$

From the conservation law

$$u_t(x,t) + \phi_x(x,t) \;=\; f(x,t)$$

we deduce the *diffusion equation* in one-dimensional case

$$u_t(x,t) - ku_{xx}(x,t) \;=\; f(x,t)$$

Transport-diffusion equation with decay and source:

$$u_t(x,t) - ku_{xx}(x,t) + c\,u_x(x,t) + \lambda u(x,t) \;=\; f(x,t)$$

Heat flow in one dimension

$u(x,t)$ - temperature

$U(x,t)$ - density of internal energy

$$U = c\varrho u$$

c - specific heat capacity, ϱ - mass density

Consider the conservation law of internal energy

$$U_t(x,t) + \phi_x(x,t) \;=\; Q(x,t)$$

where ϕ is the flux of internal energy (heat flux) and Q is the source function of internal energy

This yields

$$c\varrho u_t(x,t) + \phi_x(x,t) \;=\; Q(x,t)$$

Constitutive law (Fourier's law):

$$\phi = -K u_x.$$

We obtain the *heat equation* in one dimension

$$c\varrho u_t(x,t) - K u_{xx}(x,t) \;=\; Q(x,t)$$

Dividing by $c\varrho$ we have

$$u_t(x,t) - ku_{xx}(x,t) \;=\; f(x,t)$$

where $k = \frac{K}{c\varrho}$, $f = \frac{Q}{c\varrho}$.

This coincides with the one-dimensional *diffusion equation*.

Diffusion in multiple dimensions

Conservation law:

$$u_t(\mathbf{x},t) + \operatorname{div}\phi(\mathbf{x},t) = f(\mathbf{x},t)$$

where u is a concentration of a substance

Fick's law:

$$\phi = -k \operatorname{grad} u$$

We obtain the equation

$$u_t(\mathbf{x},t) - k \operatorname{div}\operatorname{grad} u(\mathbf{x},t) = f(\mathbf{x},t)$$

or equivalently,

$$u_t(\mathbf{x},t) - k\,\Delta u(\mathbf{x},t) = f(\mathbf{x},t)$$

This is the *diffusion equation* in multiple dimensions.

Heat equation in multiple dimensions:

$$c\varrho u_t(\mathbf{x}, t) - K\Delta u(\mathbf{x}, t) \;=\; Q(\mathbf{x}, t)$$

$u(\mathbf{x}, t)$ - temperature

c - specific heat capacity, ϱ - mass density

Q - density rate of heat sources

Dividing by $c\varrho$ we have

$$u_t(\mathbf{x}, t) - k\Delta u(\mathbf{x}, t) \;=\; f(\mathbf{x}, t),$$

where $k = \frac{K}{c\varrho}$, $f = \frac{F}{c\varrho}$.

This coincides with the *multidimensional diffusion equation*

Steady state processes

Consider a steady state (time-independent) process. Then f and u do not depend on time. We have $u_t = 0$.

From the diffusion equation

$$u_t(\mathbf{x}, t) - k\Delta u(\mathbf{x}, t) \;=\; f(\mathbf{x}, t)$$

we obtain the *Poisson equation*

$$\Delta u(\mathbf{x}) \;=\; \psi(\mathbf{x})$$

where $\psi = -\frac{f}{k}$.

In the particular case $\psi = 0$ the Poisson equation has the from

$$\Delta u(\mathbf{x}) \;=\; 0$$

It is called the *Laplace equation.*

Solutions of the Laplace equation are called *harmonic functions.*

In one-dimensional case the Poisson equation is

$$u''(x) = \psi(x)$$

Equation of vibrating string (one-dimensional wave equation)

A horizontal tensioned string between points $x = 0$ and $x = l$.

Consider only vertical movements.

$u(x, t)$ - displacement

$T(x, t)$ - tension

$\varphi(x, t)$ - angle of the string

The horizontal and vertical components of the tension at the point x are

$$T(x, t) \cos \varphi(x, t) \quad \text{and} \quad T(x, t) \sin \varphi(x, t),$$

respectively.

Since there is no horizontal movement,

$$T(x, t) \cos \varphi(x, t) \quad \text{does not depend on } x$$

We note that

$$\tan \varphi(x, t) = u_x(x, t)$$

$\rho(x, t)$ - linear mass density of the string

$[a, b] \subset [0, l]$ - an arbitrary segment of the interval $[0, l]$.

Choose some point $x \in [a, b]$ and consider a small piece of the string dl on the interval $[x, x + dx]$
at time moment t.

The pulling force at the left end-point of dl is $T(x, t)$.

The pulling force at the right end-point of dl is
$T(x + dx, t)$.

The total horizontal force acting on dl is zero.

The total vertical force acting on dl is a difference of vercital pulling forces at the endpoints:

$$dF = T(x + dx, t) \sin \varphi(x + dx, t) - T(x, t) \sin \varphi(x, t) =$$
$$= [T(x, t) \sin \varphi(x, t)]_x \, dx$$

The length of dl is

$$dl = \sqrt{dx^2 + (\tan\varphi(x,t))^2 dx^2} = \sqrt{1 + (u_x(x,t))^2} dx$$

The mass of dl is

$$dm = \rho(x,t) dl = \rho(x,t)\sqrt{1 + (u_x(x,t))^2} dx$$

Since there is no horizontal movement, the mass of dl must be conserved, i.e

$$dm = \rho_0(x) dx$$

where

$$\rho_0(x) = \rho(x,t)\sqrt{1 + (u_x(x,t))^2}$$

is independent of t.

$\rho_0(x)$ is the linear density of the string at x in the case of equilibrium.

The velocity of dl is

$$v(x,t) = u_t(x,t)$$

Plugging the deduced relations into Newton's 2nd law ($(dm\, v)_t = dF$) we obtain

$$\rho_0(x) u_{tt}(x,t) dx = \left[T(x,t)\sin\varphi(x,t)\right]_x dx$$

Integrating from a to b, we arrive at the following equation for the whole segment $[a, b]$:

$$\int_a^b \rho_0(x) u_{tt}(x,t) dx = T(b,t) \sin \varphi(b,t) - T(a,t) \sin \varphi(a,t)$$

As mentioned, $T(x,t) \cos \varphi(x,t)$ does not depend on x. Therefore, we may denote

$$\tau(t) = T(x,t) \cos \varphi(x,t) \quad \text{for any } x \in [a, b]$$

We compute:

$$\begin{aligned}
&T(b,t) \sin \varphi(b,t) - T(a,t) \sin \varphi(a,t) = \\
&= T(b,t) \cos \varphi(b,t) \tan \varphi(b,t) - T(a,t) \cos \varphi(a,t) \tan \varphi(a,t) \\
&= \tau(t) \left[\tan \varphi(b,t) - \tan \varphi(a,t)\right] \\
&= \tau(t) \left[u_x(b,t) - u_x(a,t)\right] \\
&= \tau(t) \int_a^b u_{xx}(x,t) dx
\end{aligned}$$

We obtain

$$\int_a^b \rho_0(x) u_{tt}(x,t) dx = \tau(t) \int_a^b u_{xx}(x,t) dx$$

Consequently,

$$\int_a^b [\rho_0(x)u_{tt}(x,t) - \tau(t)u_{xx}(x,t)]\, dx = 0$$

Since the segment $[a, b]$ is arbitrary, we obtain the following *equation of motion*:

$$\rho_0(x)u_{tt}(x,t) = \tau(t)u_{xx}(x,t)$$

Simplifications:

1) String consists of homogeneous material, thus linear density in the equilibrium state is constant:

$\rho_0(x) \equiv \rho_0$

2) Oscillations are relatively small. This implies

$T(x,t) \approx \tau_0$, where τ_0 is the tension in the equilibrium state and

$\varphi(x,t) \approx 0$.

Thus

$\tau(t) = T(x,t)\cos\varphi(x,t) \approx \tau_0 \cos 0 = \tau_0$.

The equation takes the form

$$u_{tt}(x,t) = c^2 u_{xx}(x,t)$$

where $c = \sqrt{\frac{\tau_0}{\rho_0}}$.

Generalizations

In case of presence of external forces, the equation has the form:

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) + f(x,t)$$

where f is the density of the forces.

External force may be caused by a resistance of an environment.

In case of elastic environment, $f = -ku$, where $k > 0$ is a constant.

Then the equation has the form

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) - ku(x,t)$$

In case of external damping, $f = -ru_t$, where $r > 0$ is a constant.

Then the equation has the form

$$u_{tt}(x,t) + ru_t(x,t) = c^2 u_{xx}(x,t)$$

Steady state case

f and u do not depend on time.

Then $u_{tt} \equiv 0$. We have the Poisson equation

$$u''(x) = \psi(x)$$

where $\psi = -\frac{f}{c^2}$.

Equation of vibrating membrane

A horizontal tensioned membrane

Consider only vertical movements.

$u(\mathbf{x}, t)$ - displacement, $\mathbf{x} = (x_1, x_2)$

The equation of free motion:

$$u_{tt}(\mathbf{x}, t) = c^2 \Delta u(\mathbf{x}, t)$$

The equation of forced motion:

$$u_{tt}(\mathbf{x}, t) = c^2 \Delta u(\mathbf{x}, t) + f(\mathbf{x}, t)$$

where f is the density of external forces.

Steady state case

f and u do not depend on time.

Then $u_{tt} \equiv 0$. We have the Poisson equation

$$\Delta u(\mathbf{x}) = \psi(\mathbf{x})$$

where $\psi = -\frac{f}{c^2}$.

Three-dimensional wave equation

$$u_{tt}(\mathbf{x}, t) = c^2 \Delta u(\mathbf{x}, t) + f(\mathbf{x}, t)\,, \quad \mathbf{x} = (x_1, x_2, x_3)$$

It describes propagation of acoustic waves under certain simplifications.

Equation of electrostatics.

Mazwell equations

$$\operatorname{rot} E = 0, \quad \operatorname{div} E = \frac{\varrho}{\epsilon}$$

E - intensity of electrostatic vector field

ϵ - constant of permittivity

ϱ - volume density of the electric charge.

$$\exists \phi \,:\, E = -\operatorname{grad} \phi$$

ϕ - electric potential

Thus

$$-\operatorname{div} \operatorname{grad} \phi = \frac{\varrho}{\epsilon} \implies \Delta \phi = -\frac{\varrho}{\epsilon}$$

We have reached a Poisson equation.