Cranck-Nicolson scheme:

$$
\begin{aligned}
&-\frac{p\tau}{2h^2}u_{i-1}^{k+1}+\left(1+\frac{p\tau}{h^2}\right)u_i^{k+1}-\frac{p\tau}{2h^2}u_{i+1}^{k+1}= \\
&=\frac{p\tau}{2h^2}u_{i-1}^{k}+\left(1-\frac{p\tau}{h^2}\right)u_i^{k}+\frac{p\tau}{2h^2}u_{i+1}^{k}u_i^k+\frac{\tau}{2}[f(x_i,t_k)+f(x_i,t_{k+1})],
\end{aligned} \tag{1}
$$

$$i=1,\ldots,n-1$$

$$u_0^{k+1}=a(t_{k+1}),\quad u_n^{k+1}=b(t_{k+1}), \tag{2}$$

for $k=0,1,\ldots,l-1$ and

$$u_i^0=\varphi(x_i),\quad i=0,\ldots,n. \tag{3}$$

Local truncation error: $O(h^2+\tau^2)$. Unconditionally stable.

Discretization of wave equation

$$u_{tt}(x,t) - p u_{xx}(x,t) = f(x,t), \quad x \in (0,L), \ \ t \in (0,T),$$

$$u(0,t) = a(t), \quad u(L,t) = b(t), \quad t \in (0,T),$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in [0,L].$$

Consistency conditions: $\varphi(0) = a(0)$, $\varphi(L) = b(0)$, $\psi(0) = a'(0)$, $\psi(L) = b'(0)$.

Method of lines:

$x_i = ih$, $h = L/n$.

$\widehat{u}_i(t) \approx u(x_i, t)$

$$\left.\begin{aligned}
&\widehat{u}_i{}''(t) = p\,\frac{\widehat{u}_{i-1}(t) + \widehat{u}_{i+1}(t) - 2\widehat{u}_i(t)}{h^2} + f(x_i,t), \quad t \in (0,T),\\
&\qquad i = 1, \ldots, n-1,\\
&\widehat{u}_0(t) = a(t), \ \ \widehat{u}_n(t) = b(t), \quad t \in (0,T),\\
&\widehat{u}_i(0) = \varphi(x_i), \quad \widehat{u}_i{}'(0) = \psi(x_i), \quad i = 0, \ldots, n.
\end{aligned}\right\} (4)$$

Local truncation error: $O(h^2)$.

Explicit scheme:

$$u_i^{k+1} = 2\left(1 - \frac{p\tau^2}{h^2}\right)u_i^k + \frac{p\tau^2}{h^2}\left(u_{i-1}^k + u_{i+1}^k\right) - u_i^{k-1} + \tau^2 f(x_i, t_k),$$

$$i = 1, \ldots, n-1$$

$$u_0^{k+1} = a(t_{k+1}), \quad u_n^{k+1} = b(t_{k+1}),$$

for $k = 0, 1, \ldots, l-1$ and

$$u_i^0 = \varphi(x_i), \quad i = 0, \ldots, n,$$

$$u_i^1 = u_i^0 + \tau\psi(x_i) + \frac{\tau^2}{2}\left(p\varphi''(x_i) + f(x_i, 0)\right), \quad i = 0, \ldots, n.$$

Here $u_i^k \approx \widehat{u}_i(t_k)$.

Local truncation error: $O(h^2 + \tau^2)$. Conditionally stable. Stability condition:

$$\frac{\sqrt{p}\tau}{h} \leq 1.$$