Let us consider set of vectors of $m$ real components
$\vec{x} = (x_1, x_2, \ldots, x_n)^T$.
Denote this set by $\mathbb{R}^n$.

Norm of vector $\vec{x} \in \mathbb{R}^n$ is a real number $\|\vec{x}\|$ satisfying the following conditions:

(I) $\|\vec{x}\| \geq 0$

(II) $\vec{x} = \vec{0}$ if and only if $\|\vec{x}\| = 0$

(III) $\|\lambda\vec{x}\| = |\lambda| \, \|\vec{x}\|$ for all $\lambda \in \mathbb{R}$

(IV) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Distance between $\vec{x}, \vec{y} \in \mathbb{R}^n$ is

$$\|\vec{x} - \vec{y}\|.$$

Examples of norms:

$$\|\vec{x}\|_2 = \left[x_1^2 + x_2^2 + \ldots + x_n^2\right]^{1/2}$$

$$\|\vec{x}\|_1 = |x_1| + |x_2| + \ldots + |x_n|$$

$$\|\vec{x}\|_\infty = \max\{|x_1|; |x_2|; \ldots; |x_n|\}$$

Norms of matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

General definition:

$$\|A\| = \max_{\vec{x} \in \mathbb{R}^n} \frac{\|A\vec{x}\|}{\|\vec{x}\|}.$$

$$\|A\|_2 = \max_{\vec{x} \in \mathbb{R}^n} \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2}, \quad \|A\|_1 = \max_{\vec{x} \in \mathbb{R}^n} \frac{\|A\vec{x}\|_1}{\|\vec{x}\|_1},$$

$$\|A\|_\infty = \max_{\vec{x} \in \mathbb{R}^n} \frac{\|A\vec{x}\|_\infty}{\|\vec{x}\|_\infty}$$

Formulas:

$$\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\|_2 = \sqrt{\lambda_{\max}},$$

where $\lambda_{max}$ is biggest eigenvalue of $A^T A$.

**Methods to solve linear systems of algebraic equations.**

$$
\begin{array}{ccccccc}
a_{11}x_1 & + & a_{12}x_2 & + \dots + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + \dots + & a_{2n}x_n & = & b_2 \\
 & & \dots & & & & \\
a_{n1}x_1 & + & a_{n2}x_2 & + \dots + & a_{nn}x_n & = & b_n
\end{array}
$$

$$A\vec{x} = \vec{b}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{pmatrix}.$$

*Direct methods*

Gaussian elimination

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & \dots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} & b_i \\ & \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{pmatrix}$$

$$\begin{pmatrix} 1 & \beta_{12} & \dots & \beta_{1n} & \theta_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & \dots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} & b_i \\ & \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{pmatrix}$$

$$\begin{pmatrix} 1 & \beta_{12} \ldots & \beta_{1n} & \theta_1 \\ a_{21} & a_{22} \ldots & a_{2n} & b_2 \\ & \ldots & & \\ a_{i1} & a_{i2} \ldots & a_{in} & b_i \\ & \ldots & & \\ a_{n1} & a_{n2} \ldots & a_{nn} & b_n \end{pmatrix}$$

New matrix:

$$\begin{pmatrix} 1 & \beta_{12} & \beta_{13} \ldots & \beta_{1n} & \theta_1 \\ 0 & a'_{22} & a'_{23} \ldots & a'_{2n} & b'_2 \\ 0 & a'_{32} & a'_{33} \ldots & a'_{3n} & b'_3 \\ & & \ldots & & \\ 0 & a'_{n2} & a'_{n3} \ldots & a'_{nn} & b'_n \end{pmatrix}$$

Here

$$a'_{ik} = a_{ik} - a_{i1}\beta_{1k} \quad (k = 2, \ldots, n), \qquad b'_i = b_i - a_{i1}\theta_1\,,$$
$$i = 2, \ldots, n.$$

$$\begin{pmatrix} 1 & \beta_{12} & \beta_{13} & \dots & \beta_{1n} & \theta_1 \\ 0 & 1 & \beta_{23} & \dots & \beta_{2n} & \theta_2 \\ 0 & 0 & a''_{33} & \dots & a''_{3n} & b''_3 \\ & & \dots & & & \\ 0 & 0 & a''_{n3} & \dots & a''_{nn} & b''_n \end{pmatrix}$$

$$\begin{pmatrix} 1 & \beta_{12} & \beta_{13} & \dots & \beta_{1n} & \theta_1 \\ 0 & 1 & \beta_{23} & \dots & \beta_{2n} & \theta_2 \\ 0 & 0 & 1 & \dots & \beta_{3n} & \theta_3 \\ & & \dots & & & \\ 0 & 0 & 0 & \dots & 1 & \theta_n \end{pmatrix}$$

$$\begin{array}{rcl} x_1 + \beta_{12}x_2 + \beta_{13}x_3 + \dots + \beta_{1n}x_n & = & \theta_1 \\ x_2 + \beta_{23}x_3 + \dots + \beta_{2n}x_n & = & \theta_2 \\ \dots & & \\ x_{n-1} + \beta_{n-1,n}x_n & = & \theta_{n-1} \\ x_n & = & \theta_n. \end{array}$$

# Band matrices

$$
\begin{pmatrix}
\times & \times & & & & & & \\
\times & \times & \times & & & & & \\
 & \times & \times & \times & & & & \\
 & & \times & \times & \times & & & \\
 & & & & \ddots & & & \\
 & & & & & \times & \times & \times \\
 & & & & & & \times & \times
\end{pmatrix}
$$

$$
\begin{pmatrix}
\times & \times & \times & & & & & & & & \\
\times & \times & \times & \times & & & & & & & \\
\times & \times & \times & \times & \times & & & & & & \\
 & \times & \times & \times & \times & \times & & & & & \\
 & & \times & \times & \times & \times & \times & & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & & \times & \times & \times & \times & \times \\
 & & & & & & & \times & \times & \times & \times \\
 & & & & & & & & \times & \times & \times
\end{pmatrix}
$$

LU-factorization

$$A = LU,$$

where

$$L = \begin{pmatrix} l_{11} & 0 \dots & 0 \\ l_{21} & l_{22} \dots & 0 \\ & \dots & \\ l_{n1} & l_{n2} \dots & l_{nn} \end{pmatrix} \quad U = \begin{pmatrix} u_{11} & u_{12} \dots & u_{1n} \\ 0 & u_{22} \dots & u_{2n} \\ & \dots & \\ 0 & 0 \dots & u_{nn} \end{pmatrix}.$$

$l_{ii} = 1$ Doolittle's method

$u_{ii} = 1$ Crout's method

if $A$ is symmetric, i.e. $a_{ij} = a_{ji}$, then is is possible to take

$$U = L^T \qquad \Leftrightarrow \qquad A = LL^T$$

This is Cholesky's factorization

*Indirect methods.*

$$A\vec{x} = \vec{b}$$

Iteration: choose initial guess

$$\vec{x}^0 = (x_1^0, \ldots, x_n^0)^T.$$

Compute successively

$$\vec{x}^1 = (x_1^1, \ldots, x_n^1)^T$$

$$\vec{x}^2 = (x_1^2, \ldots, x_n^2)^T$$

and so on

Stopping criterion:

$$\|\vec{x}^k - \vec{x}^{k-1}\| < \varepsilon$$

$$
\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \ldots & + & a_{2n}x_n & = & b_3 \\
 & & \ldots & & & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{nn}x_n & = & b_n.
\end{array}
$$

Solving for main diagonal:

$$
\begin{array}{ccccccccccccc}
x_1 & + & \frac{a_{12}}{a_{11}}x_2 & + & \frac{a_{13}}{a_{11}}x_3 & + & \ldots & + & \frac{a_{1,n-1}}{a_{11}}x_{m-1} & + & \frac{a_{1n}}{a_{11}}x_n & = & \frac{b_1}{a_{11}} \\
\frac{a_{21}}{a_{22}}x_1 & + & x_2 & + & \frac{a_{23}}{a_{22}}x_3 & + & \ldots & + & \frac{a_{2,n-1}}{a_{22}}x_{n-1} & + & \frac{a_{2n}}{a_{22}}x_n & = & \frac{b_2}{a_{22}} \\
\frac{a_{31}}{a_{33}}x_1 & + & \frac{a_{32}}{a_{33}}x_2 & + & x_3 & + & \ldots & + & \frac{a_{3,n-1}}{a_{33}}x_{n-1} & + & \frac{a_{3n}}{a_{33}}x_n & = & \frac{b_3}{a_{33}} \\
 & & & & \ldots & & & & & & & & \\
\frac{a_{n1}}{a_{nn}}x_1 & + & \frac{a_{n2}}{a_{nn}}x_2 & + & \frac{a_{n3}}{a_{nn}}x_3 & + & \ldots & + & \frac{a_{n,n-1}}{a_{nn}}x_{n-1} & + & x_n & = & \frac{b_n}{a_{nn}}.
\end{array}
$$

$$
\begin{array}{lllllll}
x_1 & = & & - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 & - \ldots - & \frac{a_{1,n-1}}{a_{11}}x_{n-1} - \frac{a_{1n}}{a_{11}}x_n & + \frac{b_1}{a_{11}} \\
x_2 & = & - \frac{a_{21}}{a_{22}}x_1 & - \frac{a_{23}}{a_{22}}x_3 & - \ldots - & \frac{a_{2,n-1}}{a_{22}}x_{n-1} - \frac{a_{2n}}{a_{22}}x_n & + \frac{b_2}{a_{22}} \\
x_3 & = & - \frac{a_{31}}{a_{33}}x_1 & - \frac{a_{32}}{a_{33}}x_2 & - \ldots - & \frac{a_{3,n-1}}{a_{33}}x_{n-1} - \frac{a_{3n}}{a_{33}}x_n & + \frac{b_3}{a_{33}} \\
 & & \ldots & & & & \\
x_n & = & - \frac{a_{n1}}{a_{nn}}x_1 & - \frac{a_{n2}}{a_{nn}}x_2 - \frac{a_{n3}}{a_{nn}}x_3 & - \ldots - & \frac{a_{n,n-1}}{a_{nn}}x_{n-1} & + \frac{b_n}{a_{nn}}.
\end{array}
$$

$$
\begin{aligned}
x_1 &= \quad - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - \ldots - \frac{a_{1,n-1}}{a_{11}}x_{n-1} - \frac{a_{1n}}{a_{11}}x_n + \frac{b_1}{a_{11}}\\
x_2 &= -\frac{a_{21}}{a_{22}}x_1 \quad - \frac{a_{23}}{a_{22}}x_3 - \ldots - \frac{a_{2,n-1}}{a_{22}}x_{n-1} - \frac{a_{2n}}{a_{22}}x_n + \frac{b_2}{a_{22}}\\
x_3 &= -\frac{a_{31}}{a_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2 - \ldots \quad - \frac{a_{3,n-1}}{a_{33}}x_{n-1} - \frac{a_{3n}}{a_{33}}x_n + \frac{b_3}{a_{33}}\\
&\ldots\\
x_n &= -\frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}}{a_{nn}}x_2 - \frac{a_{n3}}{a_{nn}}x_3 - \ldots - \frac{a_{n,n-1}}{a_{nn}}x_{n-1} \quad + \frac{b_n}{a_{nn}}.
\end{aligned}
$$

Jacobi iteration:

$$
\begin{aligned}
x_1^k &= \quad - \frac{a_{12}}{a_{11}}x_2^{k-1} - \frac{a_{13}}{a_{11}}x_3^{k-1} - \ldots - \frac{a_{1,n-1}}{a_{11}}x_{n-1}^{k-1} - \frac{a_{1n}}{a_{11}}x_n^{k-1} + \frac{b_1}{a_{11}}\\
x_2^k &= -\frac{a_{21}}{a_{22}}x_1^{k-1} \quad - \frac{a_{23}}{a_{22}}x_3^{k-1} - \ldots - \frac{a_{2,n-1}}{a_{22}}x_{n-1}^{k-1} - \frac{a_{2n}}{a_{22}}x_n^{k-1} + \frac{b_2}{a_{22}}\\
x_3^k &= -\frac{a_{31}}{a_{33}}x_1^{k-1} - \frac{a_{32}}{a_{33}}x_2^{k-1} - \ldots \quad - \frac{a_{3,n-1}}{a_{33}}x_{n-1}^{k-1} - \frac{a_{3n}}{a_{33}}x_n^{k-1} + \frac{b_3}{a_{33}}\\
&\ldots\\
x_n^k &= -\frac{a_{n1}}{a_{nn}}x_1^{k-1} - \frac{a_{n2}}{a_{nn}}x_2^{k-1} - \frac{a_{n3}}{a_{nn}}x_3^{k-1} - \ldots - \frac{a_{n,n-1}}{a_{nn}}x_{n-1}^{k-1} \quad + \frac{b_n}{a_{nn}}.
\end{aligned}
$$

Gauss-Seidel iteration:

$$
\begin{aligned}
x_1^k &= \quad - \frac{a_{12}}{a_{11}}x_2^{k-1} - \frac{a_{13}}{a_{11}}x_3^{k-1} - \ldots - \frac{a_{1,n-1}}{a_{11}}x_{n-1}^{k-1} - \frac{a_{1n}}{a_{11}}x_n^{k-1} + \frac{b_1}{a_{11}}\\
x_2^k &= -\frac{a_{21}}{a_{22}}x_1^{k} \quad - \frac{a_{23}}{a_{22}}x_3^{k-1} - \ldots - \frac{a_{2,n-1}}{a_{22}}x_{n-1}^{k-1} - \frac{a_{2n}}{a_{22}}x_n^{k-1} + \frac{b_2}{a_{22}}\\
x_3^k &= -\frac{a_{31}}{a_{33}}x_1^{k} - \frac{a_{32}}{a_{33}}x_2^{k} - \ldots \quad - \frac{a_{3,n-1}}{a_{33}}x_{n-1}^{k-1} - \frac{a_{3n}}{a_{33}}x_n^{k-1} + \frac{b_3}{a_{33}}\\
&\ldots\\
x_n^k &= -\frac{a_{n1}}{a_{nn}}x_1^{k} - \frac{a_{n2}}{a_{nn}}x_2^{k} - \frac{a_{n3}}{a_{nn}}x_3^{k} - \ldots - \frac{a_{n,n-1}}{a_{nn}}x_{n-1}^{k} \quad + \frac{b_n}{a_{nn}}.
\end{aligned}
$$

Convergence

$\|\vec{x}^k - \vec{x}^*\| \to 0 \quad \text{as} \;\; k \to \infty \quad \text{where} \quad \vec{x}^*$ − exact solution

Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

*is strictly diagonally dominant*, if for any $i = 1, \dots, n$ the inequality

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|.$$

is valid.

If $A$ is diagonally dominant, then Jacobi and Gauss-Seidel iterations converge in case of any initial guess.

Relaxation method.

Formula for $i$-th row in Gauss-Seidel iteration:

$$x_i^k = \frac{1}{a_{ii}} \left[ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^k - \sum_{j=i+1}^{n} a_{ij} x_j^{k-1} \right]$$

Stepsize of going from $x_i^{k-1}$ to $x_i^k$:

$$x_i^k - x_i^{k-1} = \frac{1}{a_{ii}} \left[ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^k - \sum_{j=i}^{n} a_{ij} x_j^{k-1} \right]$$

Change this stepsize by factor $\omega > 0$. Here $\omega$ is so-called relaxation parameter. We obtain

$$x_i^k - x_i^{k-1} = \frac{\omega}{a_{ii}} \left[ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^k - \sum_{j=i}^{n} a_{ij} x_j^{k-1} \right]$$

Express again $x_i^k$:

$$x_i^k = (1-\omega) x_i^{k-1} + \frac{\omega}{a_{ii}} \left[ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^k - \sum_{j=i+1}^{n} a_{ij} x_j^{k-1} \right].$$

$0 < \omega < 1$ - underrelaxation

$\omega > 1$ - overrelaxation

**Formulas of numerical differentiation**.

Taylor's formula of $k+1$-times continuously differentiable function $u$ at $x_i$:

$$u(x) = \underbrace{u(x_i) + u'(x_i)(x - x_i) + \frac{u''(x_i)}{2!}(x - x_i)^2 + \ldots + \frac{u^{(k)}(x_i)}{k!}(x - x_i)^k}_{\text{Taylor's polynomial}} +$$

$$+ \underbrace{\frac{u^{(k+1)}(\xi)}{(k+1)!}(x - x_i)^{k+1}}_{\text{remainder}}.$$

Forward difference formula:

$$u'(x_i) = \frac{u(x_{i+1}) - u(x_i)}{h} - \frac{u''(\xi)}{2}h\,, \quad \xi \in (x_i, x_{i+1}) \quad (1)$$

$$u'(x_i) \approx \frac{u(x_{i+1}) - u(x_i)}{h} \quad (2)$$

Backward difference formula:

$$u'(x_i) = \frac{u(x_i) - u(x_{i-1})}{h} + \frac{u''(\xi)}{2}h\,, \quad \xi \in (x_{i-1}, x_i) \quad (3)$$

$$u'(x_i) \approx \frac{u(x_i) - u(x_{i-1})}{h} \quad (4)$$

Symmetric difference formula:

$$u'(x_i) = \frac{u(x_{i+1}) - u(x_{i-1})}{2h} - \frac{u'''(\xi) + u'''(\eta)}{12} h^2, \quad (5)$$

$$\xi \in (x_i, x_{i+1}),\ \eta \in (x_{i-1}, x_i) \quad (6)$$

$$u'(x_i) \approx \frac{u(x_{i+1}) - u(x_{i-1})}{2h} \quad (7)$$

Difference formula for 2nd order derivative:

$$u''(x_i) = \frac{u(x_{i-1}) + u(x_{i+1}) - 2u(x_i)}{h^2} - \frac{u^{iv}(\xi) + u^{iv}(\eta)}{24} h^2, \quad (8)$$

$$\xi \in (x_i, x_{i+1}),\ \eta \in (x_{i-1}, x_i)$$

$$u''(x_i) \approx \frac{u(x_{i-1}) + u(x_{i+1}) - 2u(x_i)}{h^2} \quad (9)$$

**FDM for 1D problem**

$$-pu''(x) + qu(x) = f(x), \quad x \in (0, L),$$
$$u(0) = a, \quad u'(L) = b.$$

$$x_i = ih,\ i = 0, \ldots, n, \qquad h = L/n.$$

$$-\frac{p}{h^2}u_{i-1} + \left(\frac{2p}{h^2} + q\right)u_i - \frac{p}{h^2}u_{i+1} = f(x_i),\ i = 1, \ldots, n-1, \tag{10}$$

$$u_0 = a, \tag{11}$$

$$-\frac{u_{n-1}}{2h} + \frac{u_{n+1}}{2h} = b, \tag{12}$$

$$-\frac{p}{h^2}u_{n-1} + \left(\frac{2p}{h^2} + q\right)u_n - \frac{p}{h^2}u_{n+1} = f(x_n). \tag{13}$$

Here $u_i \approx u(x_i)$, $i = 0, \ldots, n+1$.

Additional gridpoint: $x_{n+1} = L + h$.