FVM for 1D problem

$$\int_0^L \left(-(pu_h')' + qu_h - f\right) 1_{\omega_i} dx = 0, \quad i = 1, \ldots, n, \quad (1)$$

$$1_{\omega_i}(x) = \begin{cases} 1 & \text{for } x \in \omega_i, \\ 0 & \text{for } x \notin \omega_i \end{cases}$$

$$\omega_0 = (x_0, x_{\frac{1}{2}}), \quad \omega_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}),\ i = 1, \ldots, n-1, \quad \omega_n = (x_{n-\frac{1}{2}}, x_n).$$

$$x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$$

$$\int_{x_0}^{x_{\frac{1}{2}}} (-(pu_h')' + qu_h - f) dx = 0,$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (-(pu_h')' + qu_h - f) dx = 0, \quad i = 1, \ldots, n-1,$$

$$\int_{x_{n-\frac{1}{2}}}^{x_n} (-(pu_h')' + qu_h - f) dx = 0.$$

Basic equations of FVM:

$$-(pu_h)'(x_{\frac{1}{2}}) + (pu_h')(x_0) + \int_{x_0}^{x_{\frac{1}{2}}} (qu_h - f)dx = 0, \qquad (2)$$

$$-(pu_h')(x_{i+\frac{1}{2}}) + (pu_h')(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0, \qquad (3)$$

$$i = 1, \ldots, n-1,$$

$$-(pu_h')(x_n) + (pu_h')(x_{n-\frac{1}{2}}) + \int_{x_{n-\frac{1}{2}}}^{x_n} (qu_h - f)dx = 0. \qquad (4)$$

$$u_h = \sum_{j=0}^{n} u_j \varphi_j \qquad (5)$$

φ_j - the same shape functions as in FEM.

A system of equations of FVM depends on boundary conditions.

Boundary conditions: $u(0) = a$, $u'(L) = b$.
Solution is searched in the form

$$u_h = a\varphi_0 + \sum_{j=1}^{n} u_j \varphi_j \tag{6}$$

and the system of equations is

$$-(pu_h')(x_{i+\frac{1}{2}}) + (pu_h')(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0, \tag{7}$$

$$i = 1, \ldots, n-1,$$

$$-p(x_n)b + (pu_h')(x_{n-\frac{1}{2}}) + \int_{x_{n-\frac{1}{2}}}^{x_n} (qu_h - f)dx = 0. \tag{8}$$

Boundary conditions: $u(0) = a$, $u(L) = b$.
Solution is searched in the form

$$u_h = a\varphi_0 + \sum_{j=1}^{n-1} u_j \varphi_j + b\varphi_n \tag{9}$$

and the system of equations is

$$-(pu_h')(x_{i+\frac{1}{2}}) + (pu_h')(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0, \tag{10}$$

$$i = 1, \ldots, n-1.$$

Boundary conditions: $u'(0) = a$, $u'(L) = b$.
Solution is searched in the form

$$u_h = \sum_{j=0}^{n} u_j \varphi_j \tag{11}$$

and the system of equations is

$$-(pu_h)'(x_{\frac{1}{2}}) + p(x_0)a + \int_{x_0}^{x_{\frac{1}{2}}} (qu_h - f)dx = 0, \tag{12}$$

$$-(pu_h')(x_{i+\frac{1}{2}}) + (pu_h')(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0, \tag{13}$$

$$i = 1, \ldots, n-1,$$

$$-p(x_n)b + (pu_h')(x_{n-\frac{1}{2}}) + \int_{x_{n-\frac{1}{2}}}^{x_n} (qu_h - f)dx = 0. \tag{14}$$

Auxiliary formulas:

$$u_h'(x_{i-\frac{1}{2}}) = \frac{1}{h}(u_i - u_{i-1}), \quad u_h'(x_{i+\frac{1}{2}}) = \frac{1}{h}(u_{i+1} - u_i) \tag{15}$$

By means of the rectangular rule

$$\int_{x_0}^{x_{\frac{1}{2}}} w(x)dx \approx (x_{\frac{1}{2}} - x_0)w(x_n) = \frac{h}{2}w(x_0), \tag{16}$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x)dx \approx (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}})w(x_i) = hw(x_i), \tag{17}$$

$$\int_{x_{n-\frac{1}{2}}}^{x_n} w(x)dx \approx (x_n - x_{n-\frac{1}{2}})w(x_n) = \frac{h}{2}w(x_n). \tag{18}$$

Let's return to the systems of equations.

A. Boundary conditions: $u(0) = a$, $u'(L) = b$.

Solution is searched in the form

$$u_h = a\varphi_0 + \sum_{j=1}^{n} u_j \varphi_j \tag{19}$$

and the system of equations is

$$-(pu_h')(x_{i+\frac{1}{2}}) + (pu_h')(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0,$$

$$i = 1, \ldots, n-1,$$

$$-p(x_n)b + (pu_h')(x_{n-\frac{1}{2}}) + \int_{x_{n-\frac{1}{2}}}^{x_n} (qu_h - f)dx = 0.$$

Using the formula of u_h and formulas for derivatives in nodes we have

$$-p(x_{\frac{3}{2}})\frac{1}{h}(u_2 - u_1) + p(x_{\frac{1}{2}})\frac{1}{h}(u_1 - a) + \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} \Big[q(a\varphi_0 + \sum_{j=1}^{n} u_j\varphi_j) - f\Big]dx = 0,$$

$$-p(x_{i+\frac{1}{2}})\frac{1}{h}(u_{i+1} - u_i) + p(x_{i-\frac{1}{2}})\frac{1}{h}(u_i - u_{i-1}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Big[q(a\varphi_0 + \sum_{j=1}^{n} u_j\varphi_j) - f\Big]dx = 0,$$

$$i = 2, \ldots, n-1,$$

$$-p(x_n)b + p(x_{n-\frac{1}{2}})\frac{1}{h}(u_n - u_{n-1}) + \int_{x_{n-\frac{1}{2}}}^{x_n} \Big[q(a\varphi_0 + \sum_{j=1}^{n} u_j\varphi_j) - f\Big]dx = 0.$$

This results in the system

$$\frac{p(x_{\frac{1}{2}})+p(x_{\frac{3}{2}})}{h}u_1-\frac{p(x_{\frac{3}{2}})}{h}u_2+\sum_{j=1}^{n}u_j\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}q\varphi_j dx=$$

$$=\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}(f-aq\varphi_0)dx+\frac{p(x_{\frac{1}{2}})}{h}a, \tag{20}$$

$$-\frac{p(x_{i-\frac{1}{2}})}{h}u_{i-1}+\frac{p(x_{i-\frac{1}{2}})+p(x_{i+\frac{1}{2}})}{h}u_i-\frac{p(x_{i+\frac{1}{2}})}{h}u_{i+1}+\sum_{j=1}^{n}u_j\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}q\varphi_j dx=$$

$$=\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}f dx,\quad i=2,\dots,n-1, \tag{21}$$

$$-\frac{p(x_{n-\frac{1}{2}})}{h}u_{n-1}+\frac{p(x_{n-\frac{1}{2}})}{h}u_n+\sum_{j=1}^{n}u_j\int_{x_{n-\frac{1}{2}}}^{x_n}q\varphi_j dx=$$

$$=\int_{x_{n-\frac{1}{2}}}^{x_n}f dx+p(x_n)b \tag{22}$$

B. Boundary conditions: $u(0) = a$, $u(L) = b$.
Solution is searched in the form

$$u_h = a\varphi_0 + \sum_{j=1}^{n-1} u_j \varphi_j + b\varphi_n \tag{23}$$

and the system of equations is

$$-(pu_h')(x_{i+\frac{1}{2}}) + (pu_h')(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0,$$

$$i = 1, \ldots, n-1.$$

The system for $u_1, \dots, u_{n-1}$:

$$\frac{p(x_{\frac{1}{2}}) + p(x_{\frac{3}{2}})}{h} u_1 - \frac{p(x_{\frac{3}{2}})}{h} u_2 + \sum_{j=1}^{n-1} u_j \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} q\varphi_j dx =$$
$$= \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} (f - aq\varphi_0) dx + \frac{p(x_{\frac{1}{2}})}{h} a, \tag{24}$$

$$-\frac{p(x_{i-\frac{1}{2}})}{h} u_{i-1} + \frac{p(x_{i-\frac{1}{2}}) + p(x_{i+\frac{1}{2}})}{h} u_i - \frac{p(x_{i+\frac{1}{2}})}{h} u_{i+1} + \sum_{j=1}^{n-1} u_j \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q\varphi_j dx =$$
$$= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f dx, \quad i = 2, \dots, n-2, \tag{25}$$

$$-\frac{p(x_{n-1-\frac{1}{2}})}{h} u_{n-2} + \frac{p(x_{n-1-\frac{1}{2}}) + p(x_{n-1+\frac{1}{2}})}{h} u_{n-1} + \sum_{j=1}^{n-1} u_j \int_{x_{n-1-\frac{1}{2}}}^{x_{n-1+\frac{1}{2}}} q\varphi_j dx =$$
$$= \int_{x_{n-1-\frac{1}{2}}}^{x_{n-1+\frac{1}{2}}} (f - aq\varphi_n) dx + \frac{p(x_{n-1+\frac{1}{2}})}{h} b \tag{26}$$

C. Boundary conditions: $u'(0) = a$, $u'(L) = b$.
Solution is searched in the form

$$u_h = \sum_{j=0}^{n} u_j \varphi_j \tag{27}$$

and the system of equations is

$$-(pu_h)'(x_{\frac{1}{2}}) + p(x_0)a + \int_{x_0}^{x_{\frac{1}{2}}} (qu_h - f)dx = 0,$$

$$-(pu_h')(x_{i+\frac{1}{2}}) + (pu_h')(x_{i-\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (qu_h - f)dx = 0,$$

$$i = 1, \ldots, n-1,$$

$$-p(x_n)b + (pu_h')(x_{n-\frac{1}{2}}) + \int_{x_{n-\frac{1}{2}}}^{x_n} (qu_h - f)dx = 0.$$

The system for $u_0, \ldots, u_n$:

$$\frac{p(x_{\frac{1}{2}})}{h}u_0 - \frac{p(x_{\frac{1}{2}})}{h}u_1 + \sum_{j=0}^{n} u_j \int_{x_0}^{x_{\frac{1}{2}}} q\varphi_j dx =$$

$$= \int_{x_0}^{x_{\frac{1}{2}}} f dx - p(x_0)a, \tag{28}$$

$$-\frac{p(x_{i-\frac{1}{2}})}{h}u_{i-1} + \frac{p(x_{i-\frac{1}{2}}) + p(x_{i+\frac{1}{2}})}{h}u_i - \frac{p(x_{i+\frac{1}{2}})}{h}u_{i+1} + \sum_{j=0}^{n} u_j \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q\varphi_j dx =$$

$$= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f dx, \quad i = 1, \ldots, n-1, \tag{29}$$

$$-\frac{p(x_{n-\frac{1}{2}})}{h}u_{n-1} + \frac{p(x_{n-\frac{1}{2}})}{h}u_n + \sum_{j=0}^{n} u_j \int_{x_{n-\frac{1}{2}}}^{x_n} q\varphi_j dx =$$

$$= \int_{x_{n-\frac{1}{2}}}^{x_n} f dx + p(x_n)b \tag{30}$$

FDM for 2D problem

$$-\Delta u + qu = f \quad \text{in} \quad (0, L_x) \times (0, L_y) \qquad (31)$$

Grid:

$x_i = ih_x$, $i = 0, \ldots, n$, $x_n = L_x$,
$y_j = jh_y$, $j = 0, \ldots, m$, $y_m = L_y$.

$u_{ij} \approx u(x_i, y_j)$

5-point difference scheme

$$\left(\frac{2}{h_x^2} + \frac{2}{h_y^2} + q\right) u_{ij} \qquad (32)$$

$$- \frac{1}{h_x^2} u_{i-1,j} - \frac{1}{h_x^2} u_{i+1,j} - \frac{1}{h_y^2} u_{i,j-1} - \frac{1}{h_y^2} u_{i,j+1} = f(x_i, y_j),$$

$$i = 1, \ldots, n-1, \; j = 1, \ldots, m-1.$$

A. First kind boundary conditions

$$u(x,0) = g_1(x), \;\; u(x, L_y) = g_2(x),$$
$$u(0,y) = g_3(y), \;\; u(L_x, y) = g_4(y).$$

Equations corresponding to these conditions:

$$\begin{aligned} &u_{i0} = g_1(x_i),\, u_{im} = g_2(x_i),\, i = 0, \ldots, n, \\ &u_{0j} = g_3(y_j),\, u_{nj} = g_4(y_j),\, j = 0, \ldots, m. \end{aligned} \tag{33}$$

B. Mixed type boundary conditions

$$u(x,0) = g_1(x), \quad u(x, L_y) = g_2(x),$$
$$u(0,y) = g_3(y), \tag{34}$$

$$u_x(L_x, y) = \gamma(y). \tag{35}$$

Equations corresponding to conditions (34):

$$u_{i0} = g_1(x_i),\ u_{im} = g_2(x_i),\ i = 0, \ldots, n,$$
$$u_{0j} = g_3(y_j),\ j = 0, \ldots, m. \tag{36}$$

Equations corresponding to condition (35):

$$u_{n+1,j} - u_{n-1,j} = 2h_x\,\gamma(y_j),\ j = 1, \ldots, m-1. \tag{37}$$

Here $u_{n+1,j} \approx u(x_{n+1}, y_j)$, $x_{n+1} = L_x + h$.

Additional main equations at the right boundary:

$$\left(\frac{2}{h_x^2} + \frac{2}{h_y^2} + q\right)u_{nj} \tag{38}$$
$$-\frac{1}{h_x^2}u_{n-1,j} - \frac{1}{h_x^2}u_{n+1,j} - \frac{1}{h_y^2}u_{n,j-1} - \frac{1}{h_y^2}u_{n,j+1} = f(x_n, y_j),$$
$$j = 1, \ldots, m-1.$$