Numerical Methods II. Problems solved in class

1. Solve

$$-u''(x) + 3u(x) = 3 - 8\sin x, \quad x \in (0, \pi),$$
$$u(0) = 1, \quad u'(\pi) = 2$$

by means of the finite difference method with number of subintervals $n = 100$.

```
clear
L=pi;
a=1;
b=2;
p=1;
q=3;
f=@(x)3-8*sin(x);
n=100;
%%%
h=L/n;
x=0:h:L+h;
A=zeros(n+2,n+2);
A(1,1)=1;
y(1)=a;
for i=2:n+1
    A(i,i-1)=-p/h^2;
    A(i,i)=2*p/h^2+q;
    A(i,i+1)=-p/h^2;
    y(i)=f(x(i));
end
A(n+2,n)=-1/(2*h);
A(n+2,n+2)=1/(2*h);
y(n+2)=b;
%%%%
u=A\y';
```

```
plot(x(1:n+1),u(1:n+1))
xlabel('x')
ylabel('u')
grid on
```

2. Solve

$$-u''(x) + 2u(x) = x, \quad x \in (0,1),$$
$$u'(0) = u'(1) = 0$$

by means of the finite difference method with number of subintervals $n = 50$.

```
clear
L=1;
a=0;
b=0;
p=1;
q=2;
f=@(x)x;
n=50;
%%%
h=L/n;
x=-h:h:L+h;
A=zeros(n+3,n+3);
A(1,1)=-1/(2*h);
A(1,3)=1/(2*h);
y(1)=a;
for i=2:n+2
    A(i,i-1)=-p/h^2;
    A(i,i)=2*p/h^2+q;
    A(i,i+1)=-p/h^2;
    y(i)=f(x(i));
end
A(n+3,n+1)=-1/(2*h);
A(n+3,n+3)=1/(2*h);
y(n+3)=b;
%%%%
u=A\y';
plot(x(2:n+2),u(2:n+2))
xlabel('x')
ylabel('u')
grid on
```

3. Solve

$$-u''(x) = \cos x\,, \; x \in (0,\pi)\,,$$
$$u(0) = 2,\; u'(\pi) = 1,$$

by means of the Galerkin finite element method with shape functions

$$\varphi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{h} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere,} \end{cases}$$

where $i = 0, \dots, n$, $n = 50$, $h = \frac{\pi}{n}$.

```
clear
n=50;
h=pi/n;
x=0:h:pi;
A=zeros(n,n);
A(1,1)=2/h;
A(1,2)=-1/h;
y(1)=h*cos(x(2))+2/h;
for i=2:n-1
    A(i,i-1)=-1/h;
    A(i,i)=2/h;
    A(i,i+1)=-1/h;
    y(i)=h*cos(x(i+1));
end
A(n,n)=1/h;
A(n,n-1)=-1/h;
y(n)=h/2*cos(x(n+1))+1;
%%
u=A\y';
plot(x,[2;u])
xlabel('x')
ylabel('u')
grid on
```

4. Solve

$$-u''(x) + \frac{1}{x+1}u(x) = 0\,, \; x \in (0,2)\,,$$
$$u(0) = 4, \; u(2) = 2,$$

by means of the Galerkin finite element method with shape functions

$$\varphi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{h} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere,} \end{cases}$$

where $i = 0, \dots, n$, $n = 100$, $h = \frac{2}{n}$.

```
clear
n=100;
h=2/n;
x=0:h:2;
A=zeros(n-1,n-1);
A(1,1)=2/h+h/(x(2)+1);
A(1,2)=-1/h;
y(1)=4/h;
for i=2:n-2
    A(i,i-1)=-1/h;
    A(i,i)=2/h+h/(x(i+1)+1);
    A(i,i+1)=-1/h;
    y(i)=0;
end
A(n-1,n-1)=2/h+h/(x(n)+1);
A(n-1,n-2)=-1/h;
y(n-1)=2/h;
%%
u=A\y';
plot(x,[4;u;2])
xlabel('x')
ylabel('u')
grid on
```

5. Solve

$$
\begin{aligned}
-3u''(x) + e^x u(x) &= x - 1\,,\ x \in (-1, 1)\,,\\
u'(-1) = 1,\ u'(1) &= -1,
\end{aligned}
$$

by means of the Galerkin finite element method with shape functions

$$
\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{h} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere,} \end{cases}
$$

where $i = 0, \ldots, n$, $n = 200$, $h = \frac{2}{n}$.

```
clear
n=200;
h=2/n;
x=-1:h:1;
A=zeros(n+1,n+1);
A(1,1)=3/h+h/2*exp(x(1));
A(1,2)=-3/h;
y(1)=h/2*(x(1)-1)-3;
```

```
for i=2:n
    A(i,i-1)=-3/h;
    A(i,i)=6/h+h*exp(x(i));
    A(i,i+1)=-3/h;
    y(i)=h*(x(i)-1);
end
A(n+1,n)=-3/h;
A(n+1,n+1)=3/h+h/2*exp(x(n+1));
y(n+1)=h/2*(x(n+1)-1)-3;
%%
u=A\y';
plot(x,u)
xlabel('x')
ylabel('u')
grid on
```

6. Solve

$$
\begin{aligned}
-(pu')'(x) &= 10x(2-x)\,,\ x \in (0,2)\,,\\
u(0) = u(2) &= 1,
\end{aligned}
$$

$p(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 3 & \text{if } x \in (1,2) \end{cases}$ by means of the Galerkin finite element method with shape functions

$$
\varphi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{h} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere,} \end{cases}
$$

where $i = 0, \ldots, n$, $n = 200$, $h = \frac{2}{n}$.

```
clear
n=200;
f=@(x)10*x*(2-x);
%%
h=2/n;
x=0:h:2;
A=zeros(n-1,n-1);
A(1,1)=2/h;
A(1,2)=-1/h;
y(1)=h*f(x(2))+1/h;
for i=2:99
    A(i,i-1)=-1/h;
    A(i,i)=2/h;
    A(i,i+1)=-1/h;
```

```
    y(i)=h*f(x(i+1));
end
A(100,99)=-1/h;
A(100,100)=4/h;
A(100,101)=-3/h;
y(100)=h*f(x(101));
for i=101:198
    A(i,i-1)=-3/h;
    A(i,i)=6/h;
    A(i,i+1)=-3/h;
    y(i)=h*f(x(i+1));
end
A(199,198)=-3/h;
A(199,199)=6/h;
y(199)=h*f(x(200))+3/h;
%%
u=A\y';
plot(x,[1;u;1])
xlabel('x')
ylabel('u')
grid on
```

7. Solve

$$-\Delta u(x,y) + 2u(x,y) \;=\; x - y\,,\; x \in (0,1)\,,\; y \in (0,2)\,,$$
$$u(x,0) = u(x,2) = 3\,,\; x \in (0,1),$$
$$u(0,y) = 3,\; u_x(1,y) = 1\,,\; y \in (0,2)$$

by means of the finite difference method with the stepsize $h_x = h_y = 0.02$.

```
clear
Lx=1;
Ly=2;
n=50;
m=100;
h=1/n;
f=@(x,y)x-y;
%%
x=0:h:Lx+h;
y=0:h:Ly;
%%
u=3*ones(n+2,m+1);
eta=1e-5;
norm=1;
factor=1/(4/h^2+2);
```

```
k=0;
unew=u;
while norm>eta
    for j=2:m
        for i=2:n+1
            unew(i,j)=factor*(1/h^2*(u(i-1,j)+u(i+1,j)+...
                u(i,j-1)+u(i,j+1))+f(x(i),y(j)));
        end
        unew(n+2,j)=u(n,j)+2*h;
    end
    norm=max(max(abs(unew-u)));
    u=unew;
    k=k+1;
end
k
xplot=x(1:n+1);
uplot=u(1:n+1,:);
surf(xplot,y,uplot')
xlabel('x')
ylabel('y')
zlabel('u')
```

8. Solve

$$-\Delta u(x,y) + u(x,y) \;=\; -xy\,,\;\; (x,y) \in (0,2)\times(0,1)\,,$$
$$u(x,y) = 2 + 0.5\sin(2xy) \quad \text{on the boundary of } \; (0,2)\times(0,1)$$

by means of the finite difference method with numbers of subintervals $n = m = 50$.

```
clear
Lx=2;
Ly=1;
n=50;
m=50;
hx=Lx/n;
hy=Ly/m;
f=@(x,y)-x*y;
%%
x=0:hx:Lx;
y=0:hy:Ly;
%%
for i=1:n+1
    for j=1:m+1
        u(i,j)=2+0.5*sin(2*x(i)*y(j));
    end
```

```
end
eta=1e-7;
norm=1;
factor=1/(2/hx^2+2/hy^2+1);
k=0;
unew=u;
while norm>eta
    for j=2:m
        for i=2:n
            unew(i,j)=factor*(1/hx^2*(u(i-1,j)+u(i+1,j))+...
                1/hy^2*(u(i,j-1)+u(i,j+1))+f(x(i),y(j)));
        end
    end
    norm=max(max(abs(unew-u)));
    u=unew;
    k=k+1;
end
k
surf(x,y,u')
xlabel('x')
ylabel('y')
zlabel('u')
```

9. Solve

$$u_t(x,t) - u_{xx}(x,t) \;=\; t\,,\; x \in (0,2)\,,\; t \in (0,3)\,,$$
$$u(0,t) = u(2,t) = 1\,,\; t \in (0,3),$$
$$u(x,0) = 1\,,\; x \in (0,2)$$

by means of the explicit difference scheme with the numbers of subintervals n and l in the x- and t-directions, respectively, where
1) $n = 100$, $l = 20000$,
2) $n = 100$, $l = 15000$,
3) $n = 100$, $l = 14900$.

```
clear
L=2;
T=3;
n=100;
l=20000;
h=L/n;
tau=T/l;
%%
C=2*tau/h^2
%%
```

```
x=0:h:L;
t=0:tau:T;
%%
for i=1:n+1
    u(i,1)=1;
end
for k=1:l
    u(1,k+1)=1;
    for i=2:n
        u(i,k+1)=(1-C)*u(i,k)+C/2*(u(i-1,k)+u(i+1,k))+tau*t(k);
    end
    u(n+1,k+1)=1;
end
%%
surf(x,t,u')
xlabel('x')
ylabel('t')
zlabel('u')
shading interp
```

10. Solve

$$u_t(x,t) - u_{xx}(x,t) = x^2(x-1)\cos t - (6x-2)\sin t\,,\ x \in (0,1)\,,\ t \in (0,1)\,,$$
$$u(0,t) = u(1,t) = 0\,,\ t \in (0,1),$$
$$u(x,0) = 0\,,\ x \in (0,1)$$

by means of the implicit difference scheme with the numbers of subintervals $n = 50$, $l = 50$ in the x- and t-directions, respectively. Compute the maximal error over the grid. Exact solution is $u(x,t) = x^2(x-1)\sin t$.

```
clear
L=1;
T=1;
n=50;
l=50;
h=L/n;
tau=T/l;
f=@(x,t)x^2*(x-1)*cos(t)-(6*x-2)*sin(t);
%%
x=0:h:L;
t=0:tau:T;
for i=1:n+1
    u(i,1)=0;
end
A=zeros(n+1,n+1);
```

```
A(1,1)=1;
for i=2:n
    A(i,i-1)=-tau/h^2;
    A(i,i)=1+2*tau/h^2;
    A(i,i+1)=-tau/h^2;
end
A(n+1,n+1)=1;
%%
for k=1:l
    y(1)=0;
    for i=2:n
        y(i)=u(i,k)+tau*f(x(i),t(k+1));
    end
    y(n+1)=0;
    u(:,k+1)=A\y';
end
surf(x,t,u')
xlabel('x')
ylabel('t')
zlabel('u')
%%%
for k=1:l+1
    for i=1:n+1
        ue(i,k)=x(i)^2*(x(i)-1)*sin(t(k));
    end
end
error=max(max(abs(u-ue)))
```

11. Solve

$$u_t(x,t) - u_{xx}(x,t) \;=\; x^2(x-1)\cos t - (6x-2)\sin t\,,\;\; x \in (0,1)\,,\;\; t \in (0,1)\,,$$

$$u(0,t) = u(1,t) = 0\,,\;\; t \in (0,1),$$

$$u(x,0) = 0\,,\;\; x \in (0,1)$$

by means of the Crank-Nicolson scheme with the numbers of subintervals $n = 50$, $l = 50$ in the x- and t-directions, respectively. Compute the maximal error over the grid. Exact solution is $u(x,t) = x^2(x-1)\sin t$.

```
clear
L=1;
T=1;
f=@(x,t)x^2*(x-1)*cos(t)-(6*x-2)*sin(t);
n=50;
l=50;
%%%
```

```
h=L/n;
tau=T/l;
x=0:h:L;
t=0:tau:T;
%%
for i=1:n+1
    u(i,1)=0;
end
c=tau/h^2;
A=zeros(n+1,n+1);
A(1,1)=1;
for i=2:n
    A(i,i-1)=-c/2;
    A(i,i)=1+c;
    A(i,i+1)=-c/2;
end
A(n+1,n+1)=1;
for k=1:l
    y(1)=0;
    for i=2:n
        y(i)=c/2*u(i-1,k)+(1-c)*u(i,k)+c/2*u(i+1,k)+...
           tau/2*(f(x(i),t(k))+f(x(i),t(k+1)));
    end
    y(n+1)=0;
    u(:,k+1)=A\y';
end
surf(x,t,u')
xlabel('x')
ylabel('t')
zlabel('u')
%%%
for i=1:n+1
    for k=1:l+1
        ue(i,k)=x(i)^2*(x(i)-1)*sin(t(k));
    end
end
error=max(max(abs(u-ue)))
```

12. Solve

$$u_{tt}(x,t) - u_{xx}(x,t) \;=\; 0\,,\; x \in (0,2)\,,\; t \in (0,8)\,,$$
$$u(0,t) = u(2,t) = 0\,,\; t \in (0,8),$$
$$u(x,0) = x(2-x)\,,\; u_t(x,0) = 0\,,\; x \in (0,2)$$

by means of the explicit scheme with the numbers of subintervals $n = 500$, $l = 2000$ in the x- and t-directions, respectively.

```
clear
L=2;
T=8;
phi=@(x)x*(2-x);
n=500;
l=2000;
%%
h=L/n;
tau=T/l;
x=0:h:L;
t=0:tau:T;
%%
c=tau^2/h^2;
for i=1:n+1
    u(i,1)=phi(x(i));
    u(i,2)=u(i,1)-tau^2;
end
for k=2:l
    u(1,k+1)=0;
    for i=2:n
        u(i,k+1)=2*(1-c)*u(i,k)+c*(u(i-1,k)+u(i+1,k))-u(i,k-1);
    end
    u(n+1,k+1)=0;
end
surf(x,t,u')
xlabel('x')
ylabel('t')
zlabel('u')
shading interp
```