Equivalent formulation of the equation

The equation

$$-(p(x)u'(x))' + q(x)u(x) = f(x)$$

holds in the interval $(0, L)$ if and only if

$$\int_0^L (-(pu')' + qu - f)v dx = 0 \qquad \forall v \in L_2(0, L).$$

Weighted residual method for 1D problem

Approximate solution:

$$u_h(x) = \sum_{i=1}^{n} u_i \varphi_i(x) \tag{1}$$

$\varphi_1, \ldots, \varphi_n$ - basis functions

System of equations:

$$\int_0^L \Big[-(p(x)u_h'(x))' + q(x)u_h(x) - f(x) \Big] v_i(x)dx = 0, \quad i = 1, \ldots, n. \tag{2}$$

$v_1, \ldots, v_n$ - test functions

Variational formulations of 1D problem

$$\int_0^L (-(pu')' + qu - f)v dx = 0 \qquad \forall v \in L_2(0, L).$$

Assume that the test function v is differentiable.

$$-pu'(L)v(L)+pu'(0)v(0)+\int_0^L pu'v'dx+\int_0^L (qu-f)vdx = 0.$$

Introduce the boundary conditions

$$u(0) = a, \quad u'(L) = b.$$

The boundary value $u'(0)$ is not given! Additional restriction to the test function:

$$v(0) = 0.$$

Then

$$-pbv(L) + \int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0.$$

Variational formulation of the problem with boundary conditions $u(0) = a, \quad u'(L) = b$:
Find a function u that satisfies the boundary condition $u(0) = a$ and the equation

$$-pbv(L) + \int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0. \quad (3)$$

for any test function v such that $v(0) = 0$.

Variational formulation of the problem with boundary conditions $u(0) = a, \quad u(L) = b:$
Find a function u that satisfies the boundary conditions $u(0) = a$, $u(L) = b$ and the equation

$$\int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0 \tag{4}$$

for any test function v such that $v(0) = v(L) = 0$.

Variational formulation of the problem with boundary conditions $u'(0) = a, \quad u'(L) = b:$
Find a function u that satisfies the equation

$$\begin{aligned} -pbv(L) + pav(0) + \int_0^L pu'v'dx \\ + \int_0^L (qu - f)vdx = 0 \end{aligned} \tag{5}$$

for any test function v.

Galerkin FEM for 1D problems

Firstly, we follow the variational formulation of the problem with boundary conditions $u(0) = a, \quad u'(L) = b$:
Find a function u that satisfies the boundary condition $u(0) = a$ and the equation

$$-pbv(L) + \int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0.$$

for any test function v such that $v(0) = 0$.

Shape functions $\varphi_1, \dots, \varphi_n$.

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere.} \end{cases} \tag{6}$$

Approximate solution is searched in the form

$$u_h(x) = \sum_{j=0}^{n} u_j \varphi_j(x) = a\varphi_0(x) + \sum_{j=1}^{n} u_j \varphi_j(x). \tag{7}$$

The numbers $u_1, \dots, u_n$ are to be determined.

Test functions: $\varphi_1, \dots, \varphi_n$.

$$-pb\varphi_i(L) + \int_0^L pu_h'\varphi_i'dx + \int_0^L (qu_h - f)\varphi_i dx = 0, \quad i = 1, \dots, n.$$

$$
-pb\varphi_i(L) + \int_0^L p\left[a\varphi_0' + \sum_{j=1}^{n} u_j\varphi_j'\right]\varphi_i' dx
$$
$$
+ \int_0^L \left(q\left[a\varphi_0 + \sum_{j=1}^{n} u_j\varphi_j\right] - f\right)\varphi_i dx = 0, \quad i = 1, \ldots, n.
$$

This leads to the linear system of equations

$$
\sum_{j=1}^{n} u_j \left[\int_0^L p\varphi_j'\varphi_i' dx + \int_0^L q\varphi_j\varphi_i dx\right] \tag{8}
$$
$$
= \int_0^L f\varphi_i dx + pb\varphi_i(L) - a\left[\int_0^L p\varphi_0'\varphi_i' dx + \int_0^L q\varphi_0\varphi_i dx\right],
$$
$$
i = 1, \ldots, n.
$$

Analogously we obtain systems of equations in case of the boundary conditions $u(0) = a$, $u(L) = b$.

Recall the variational formulation of this problem:

Find a function u that satisfies the boundary conditions $u(0) = a$, $u(L) = b$ and the equation

$$\int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0$$

for any test function v such that $v(0) = v(L) = 0$. Approximate solution is searched in the form

$$u_h(x) = a\varphi_0(x) + \sum_{j=1}^{n-1} u_j\varphi_j(x) + b\varphi_n(x).$$

Test functions: $\varphi_1, \ldots, \varphi_{n-1}$. We obtain the system

$$\begin{aligned}\sum_{j=1}^{n-1} u_j \left[\int_0^L p\varphi_j'\varphi_i'dx + \int_0^L q\varphi_j\varphi_i dx\right] \qquad (9)\\ = \int_0^L f\varphi_i dx - a\left[\int_0^L p\varphi_0'\varphi_i'dx + \int_0^L q\varphi_0\varphi_i dx\right] \\ -b\left[\int_0^L p\varphi_n'\varphi_i'dx + \int_0^L q\varphi_n\varphi_i dx\right], \quad i = 1, \ldots, n-1.\end{aligned}$$

Let us deduce the system for the boundary conditions $u'(0) = a$, $u'(L) = b$, too.

Variational formulation of this problem:

Find a function u that satisfies the equation

$$- pbv(L) + pav(0) + \int_0^L pu'v'dx + \int_0^L (qu - f)vdx = 0$$

for any test function v. Approximate solution is searched in the form

$$u_h(x) = \sum_{j=0}^{n} u_j \varphi_j(x).$$

Test functions: $\varphi_0, \ldots, \varphi_n$. We obtain the system

$$\sum_{j=0}^{n} u_j \left[\int_0^L p\varphi_j'\varphi_i' dx + \int_0^L q\varphi_j\varphi_i dx \right] = \int_0^L f\varphi_i dx + pb\varphi_i(L) - pa\varphi_i(0), \quad i = 0, \ldots, n. \tag{10}$$

Auxiliary formulas in case $h = x_i - x_{i-1}$ - constant.

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h} & \text{for } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{h} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere.} \end{cases}$$

$$\varphi_i'(x) = \begin{cases} \frac{1}{h} & \text{for } x \in [x_{i-1}, x_i] \\ -\frac{1}{h} & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere.} \end{cases}$$

$$\int_0^L \varphi_j' \varphi_i' dx = \begin{cases} \frac{2}{h} & \text{for } j = i \notin \{0; n\} \\ \frac{1}{h} & \text{for } j = i \in \{0; n\} \\ -\frac{1}{h} & \text{for } j = i - 1 \text{ and } j = i + 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Application of trapezoidal rule:

$$\int_{x_{i-1}}^{x_{i+1}} F(x)dx \approx \frac{h}{2} \left[F(x_{i-1}) + 2F(x_i) + F(x_{i+1}) \right] \quad \text{3-point formula}$$

$$\int_{x_{i-1}}^{x_i} F(x)dx \approx \frac{h}{2} \left[F(x_{i-1}) + F(x_i) \right] \quad \text{2-point formula}$$