Some methods for Cauchy problems for systems of ODE

$$\vec{u}\,'(t) = \vec{F}(t, \vec{u}(t)), \quad t \in (0, T), \quad \vec{u}(0) = \vec{u}^0. \qquad (1)$$

Here

$$\vec{u} = (u_1, \ldots, u_n), \quad \vec{F}(t, \vec{u}) = (F_1(t, \vec{u}), \ldots, F_n(t, \vec{u})),$$

$$\vec{u}^0 = (u_1^0, \ldots, u_n^0) \in \mathbb{R}^n.$$

Grid: $t_0 = 0$, $t_1 = \tau$, $t_2 = 2\tau$, ..., $t_k = k\tau$, ...,

$\tau > 0$ - stepsize

Forward Euler method:

$$\frac{\vec{u}(t_{k+1}) - \vec{u}(t_k)}{\tau} = \vec{F}(t_k, \vec{u}(t_k)) + O(\tau) \qquad (2)$$

$$\vec{u}^{k+1} = \vec{u}^k + \tau \vec{F}(t_k, \vec{u}^k), \qquad (3)$$

where $\vec{u}^k \approx \vec{u}(t_k)$.

Backward Euler method:

$$\frac{\vec{u}(t_{k+1}) - \vec{u}(t_k)}{\tau} = \vec{F}(t_{k+1}, \vec{u}(t_{k+1})) + O(\tau) \qquad (4)$$

$$\vec{u}^{k+1} - \tau \vec{F}(t_{k+1}, \vec{u}^{k+1}) = \vec{u}^k \qquad (5)$$

Method of trapezoidal rule:

$$\frac{\vec{u}(t_{k+1}) - \vec{u}(t_k)}{\tau} = \frac{1}{2}\Bigg[\vec{F}(t_k, \vec{u}(t_k)) + \vec{F}(t_{k+1}, \vec{u}(t_{k+1}))\Bigg] + O(\tau^2) \quad (6)$$

$$\vec{u}^{k+1} = \vec{u}^k + \frac{\tau}{2}\Bigg[\vec{F}(t_k, \vec{u}^k) + \tau\vec{F}(t_{k+1}, \vec{u}^{k+1})\Bigg]. \quad (7)$$

Discretization of heat equation

$$u_t(x,t) - pu_{xx}(x,t) = f(x,t), \qquad x \in (0,L),\ t \in (0,T), \tag{8}$$

$$u(0,t) = a(t), \quad u(L,t) = b(t), \quad t \in (0,T), \tag{9}$$

$$u(x,0) = \varphi(x), \quad x \in [0,L]. \tag{10}$$

Consistency conditions: $\varphi(0) = a(0)$, $\varphi(L) = b(0)$.

Method of lines:

$$x_i = ih, \quad h = L/n.$$

$$\widehat{u}_i(t) \approx u(x_i,t) \tag{11}$$

$$\left.\begin{aligned}
&\widehat{u}_i{}'(t) = p\,\frac{\widehat{u}_{i-1}(t) + \widehat{u}_{i+1}(t) - 2\widehat{u}_i(t)}{h^2} + f(x_i,t), \quad t \in (0,T), \\
&\qquad i = 1,\dots,n-1, \\
&\widehat{u}_0(t) = a(t), \ \ \widehat{u}_n(t) = b(t), \quad t \in (0,T), \\
&\widehat{u}_i(0) = \varphi(x_i), \quad i = 0,\dots,n.
\end{aligned}\right\} \tag{12}$$

Local truncation error: $O(h^2)$.

Explicit scheme:

$$u_i^{k+1} = \left(1 - \frac{2p\tau}{h^2}\right) u_i^k + \frac{p\tau}{h^2}\left(u_{i-1}^k + u_{i+1}^k\right) + \tau f(x_i, t_k), \quad (13)$$

$$i = 1, \ldots, n-1$$

$$u_0^{k+1} = a(t_{k+1}), \quad u_n^{k+1} = b(t_{k+1}), \quad (14)$$

for $k = 0, 1, \ldots, l-1$ and

$$u_i^0 = \varphi(x_i), \quad i = 0, \ldots, n. \quad (15)$$

Here $u_i^k \approx \widehat{u}_i(t_k) \approx u(x_i, t_k)$.

Local truncation error: $O(h^2 + \tau)$. Conditionally stable. Stability condition:

$$\frac{2p\tau}{h^2} \leq 1.$$

Implicit scheme:

$$-\frac{p\tau}{h^2}u_{i-1}^{k+1} + \left(1 + \frac{2p\tau}{h^2}\right)u_i^{k+1} - \frac{p\tau}{h^2}u_{i+1}^{k+1} = u_i^k + \tau f(x_i, t_{k+1}), \quad (16)$$

$$i = 1, \ldots, n-1$$

$$u_0^{k+1} = a(t_{k+1}), \quad u_n^{k+1} = b(t_{k+1}), \quad (17)$$

for $k = 0, 1, \ldots, l-1$ and

$$u_i^0 = \varphi(x_i), \quad i = 0, \ldots, n. \quad (18)$$

Local truncation error: $O(h^2 + \tau)$. Unconditionally stable.