8 Series

8.1 Series. Sum of series

The series is an infinite sum

$$u_1 + u_2 + \ \dots \ + u_k + \ \dots \ = \sum_{k=1}^{\infty} u_k \tag{8.1}$$

The addends in this infinite sum are called the terms of the series and u_k is called the *general term*. If we assign to k some natural number, we get the related term of the series. In (8.1) the k is called the index of summation and note that the letter we use to represent the index can be any integer variable i, j, l, m, n, $\dots$. The first index is 1 for convenience, actually it can be any integer. We can write (8.1) as

$$\sum_{k=1}^{\infty} u_k = \sum_{k=0}^{\infty} u_{k+1} = \sum_{k=2}^{\infty} u_{k-1} = \dots$$

A *number series* is the series, whose terms are numbers. In our course we consider the series of real numbers. A *functional series* is the series, whose terms are functions of the variable x, i.e. $u_k = u_k(x)$, $k = 1,\ 2,\ \dots$.

A geometric series is the series

$$a + aq + aq^2 + \ \dots \ + aq^k + \ \dots \ = \sum_{k=0}^{\infty} aq^k \tag{8.2}$$

where each successive term is produced by multiplying the previous term by a constant number q (called the *common ratio* in this context).

The harmonic series is the series

$$1 + \frac{1}{2} + \frac{1}{3} + \ \dots \ + \frac{1}{k} + \ \dots \ = \sum_{k=1}^{\infty} \frac{1}{k} \tag{8.3}$$

The sum of the first n terms

$$S_n = \sum_{k=1}^{n} u_k$$

is called the nth partial sum of the series. The partial sums

$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$

....

$$S_n = u_1 + u_2 + \ \dots \ + u_n$$

....

define the sequence of partial sums

$$S_1, \ S_2, \ \dots, \ S_n, \ \dots \tag{8.4}$$

Definition. A series (8.1) is said to *converge* or to *be convergent* when the sequence (8.4) of partial sums has a finite limit. If the limit of (8.4) is infinite or does not exist, the series is said to *diverge* or to *be divergent.* When the limit of partial sums

$$\lim_{n\to\infty} S_n = S$$

exists, it is called the *sum of the series* and one writes

$$S = \sum_{k=1}^{\infty} u_k$$

It is important not to get sequences and series confused! A sequence is a list of numbers written in a specific order while an infinite series is a limit of a sequence and hence, if it exists will be a single value.

Example 1. The sum of the first n terms, i.e. the $n-1$st partial sum of the geometric series is

$$S_{n-1} = \sum_{k=0}^{n-1} aq^k = \frac{a(1-q^n)}{1-q}$$

If $|q| < 1$, then

$$\lim_{n\to\infty} q^n = 0$$

thus,

$$\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} \frac{a(1-q^n)}{1-q} = \lim_{n\to\infty} \frac{a}{1-q} - \lim_{n\to\infty} \frac{aq^n}{1-q} = \frac{a}{1-q}$$

So, if $|q| < 1$, then the geometric series converges and the sum is

$$S = \frac{a}{1-q}$$

If $q > 1$, then

$$\lim_{n\to\infty} q^n = \infty$$

therefore,

$$\lim_{n\to\infty} S_{n-1} = \infty$$

and the geometric series is divergent If $q < -1$, then $\lim\limits_{n\to\infty} q^n$ does not exist and hence, $\lim\limits_{n\to\infty} S_{n-1}$ does not exist and the geometric series is divergent. If $q = 1$, then the $n-1$st partial sum

$$S_n = \sum_{k=0}^{n-1} aq^k = \sum_{k=0}^{n-1} a = na$$

and the limit $\lim\limits_{n\to\infty} S_{n-1} = \lim\limits_{n\to\infty} = na = \infty$. If $q = -1$, then the $S_0 = a$, $S_1 = a - a = 0$, $S_2 = a - a + a = a$, $S_3 = a - a + a - a = 0$, ... We obtain the sequence of partial sums

$$a,\ 0,\ a,\ 0,\ \ \ldots$$

which has no limit. Therefore, for $q = \pm 1$ the geometric series is divergent.

Conclusion. If $|q| < 1$, then the geometric series (8.2) converges and if $|q| \geq 1$ then the geometric series diverges.

Example 2. To find the nth partial sum S_n of the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

we use the partial fractions decomposition

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

We obtain

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)}$$
$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

The limit of this sequence, i.e. the sum of this series

$$S = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

If we ignore the first term the remaining terms will also be a series that will start at $k = 2$ instead of $k = 1$ So, we can rewrite the original series (8.1) as follows,

$$\sum_{k=1}^{\infty} u_k = u_1 + \sum_{k=2}^{\infty} u_k$$

We say that we've stripped out the first term. We could have stripped out the first two terms

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \sum_{k=3}^{\infty} u_k$$

and first any number of terms respectively,

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \ldots + u_m + \sum_{k=m+1}^{\infty} u_k = \sum_{k=1}^{m} u_k + \sum_{k=m+1}^{\infty} u_k$$

The first sum on the right side of this equality is the mth partial

$$\sum_{k=1}^{m} u_k$$

sum of series (8.1). This is a finite sum, which is always finite. Assuming that $n > m$, we can write the nth partial sum

$$\sum_{k=1}^{n} u_k = \sum_{k=1}^{m} u_k + \sum_{k=m+1}^{n} u_k$$

or

$$S_n = S_m + S_{n-m}$$

where

$$S_{n-m} = \sum_{k=m+1}^{n} u_k$$

Now, if S_n has the finite limit as $n \to \infty$, then S_{n-m} must have also the finite limit. Conversely, if S_{n-m} has the finite limit as $n \to \infty$, then adding the finite sum S_m leaves the limit finite.

Similarly, S_n has the infinite limit or does not have the limit if and only if S_{n-m} has also the infinite limit or has no limit.

Conclusion. Stripping out the finite number of terms from the beginning of the series leaves the convergent series convergent and divergent series divergent. As well, adding the finite number of terms to the beginning of the series does not make the convergent series divergent and does not make the divergent series convergent.

8.2 Necessary condition for convergence of series

Suppose that the series (8.1) converges to the sum S, i.e.

$$\lim_{n\to\infty} S_n = S$$

The nth partial sum can be written

$$S_n = \sum_{k=1}^{n} u_k = \sum_{k=1}^{n-1} u_k + u_n$$

or

$$S_n = S_{n-1} + u_n$$

hence,

$$u_n = S_n - S_{n-1}$$

The convergence of the series gives, since if $n \to \infty$ then $n - 1 \to \infty$,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

We have proved an essential theorem, so called *necessary condition for the convergence of the series.*

Theorem 1. If the series (8.1) converges, then the limit of the general term

$$\lim_{n\to\infty} u_n = 0 \tag{8.5}$$

This theorem gives us a requirement for convergence but not a guarantee of convergence. In other words, the converse is not true. If $\lim\limits_{n\to\infty} u_n = 0$ the series may actually diverge. For example, the limit of the general term of the harmonic series (8.3)

$$\lim_{k\to\infty} \frac{1}{k} = 0$$

but the harmonic series is divergent. It will be a couple of subsections before we can prove this, so at this point the reader has just to believe this and know that it's possible to prove the divergence.

In order for a series to converge the series terms must go to zero in the limit. If the series terms do not go to zero in the limit then there is no way the series can converge since this would contradict the theorem, i.e. there holds.

Conclusion (the divergence test). If $\lim\limits_{n\to\infty} u_n \neq 0$ then the series (8.1) diverges.

For example the series

$$\sum_{k=1}^{\infty} 1$$

is divergent because the limit of the constant term is that constant,

$$\lim_{k\to\infty} 1 = 1 \neq 0$$

8.3 Convergence tests of positive series

In Mathematical analysis there exist a lot of tests that give us the possibility to decide whether the series converges or diverges. In this subsection we are going to consider the positive series, i.e. the series (8.1), whose all terms are positive:

$$u_k \geq 0, \quad k = 1,\ 2,\ \ldots$$

8.3.1 Comparison test

The nth partial sum of the series (8.1) is

$$S_n = S_{n-1} + u_n$$

Since for any index n $u_n \geq 0$, then

$$S_n \geq S_{n-1}$$

that means, the sequence of partial sums of the positive series is monotonically increasing. We had the theorem in Mathematical analysis I, which stated that any bounded monotonically increasing sequence has the finite limit. So, if we have succeeded to prove that the sequence of the partial sums of the positive series is bounded, we have proved the existence of the finite limit of the sequence of partial sums, that is, we have proved the convergence of the positive series.

The sequence

$$S_1,\ S_2,\ \ldots,\ S_n,\ \ldots$$

has the finite limit means by the definition of the limit that for any $\varepsilon > 0$ there exists the index $N > 0$ such that for all $n \geq N$

$$|S_n - S| < \varepsilon$$

This inequality is identical to the inequalities

$$-\varepsilon < S_n - S < \varepsilon$$

or

$$S - \varepsilon < S_n < S + \varepsilon$$

which means the sequence is bounded. We have proved the following theorem.

Theorem 1. The positive series (8.1) is convergent if and only if the sequence of its partial sums is bounded.

Suppose that we have another positive series

$$\sum_{k=1}^{\infty} v_k \tag{8.6}$$

and we know whether it converges or diverges. For instance we know that the geometric series (8.2) converges if $|q| < 1$ and diverges if $|q| \geq 1$. We know that the harmonic series is divergent and we know that

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

is convergent.

Theorem 2 (the comparison test). 1) If for any $k = 1,\ 2,\ 3,\ \ldots$

$$u_k \leq v_k$$

then the convergence of the series (8.6) yields the convergence of the series (8.1).

2) If for any $k = 1,\ 2,\ 3,\ \ldots$

$$u_k \geq v_k$$

then the divergence of the series (8.6) yields the divergence of the series (8.1).

Proof 1) Denote the nth partial sums of the series (8.1) and (8.6) by

$$S_n = \sum_{k=1}^{n} u_k$$

and

$$\sigma_n = \sum_{k=1}^{n} v_k$$

respectively. Since for any $k = 1,\ 2,\ 3,\ \ldots\ u_k \leq v_k$, then

$$S_n \leq \sigma_n$$

By the assumption the series (8.6) is convergent hence, by Theorem 1 the sequence

$$\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_n,\ \ldots, \tag{8.7}$$

is bounded by some constant σ, i.e. $\sigma_n \le \sigma$. But then $S_n \le \sigma$, which means that the sequence of the partial sums of the series (8.1)

$$S_1,\ S_2,\ \ldots,\ S_n,\ \ldots$$

is bounded thus, by Theorem 1 the series (8.1) is convergent.

2) Next, let's assume that (8.6) is divergent. Because $v_k \ge 0$ we then know that we must have

$$\lim_{n\to\infty} \sigma_n = \infty$$

The assumption $u_k \ge v_k$ yields $S_n \ge \sigma_n$ and by the limit theorem

$$\lim_{n\to\infty} S_n \ge \lim_{n\to\infty} \sigma_n$$

which means that the sequence of the partial sums of the series (8.1)

$$S_1,\ S_2,\ \ldots,\ S_n,\ \ldots$$

has no finite limit or the series (8.1) is divergent.

Example 1. Prove that the series

$$1 + \frac{1}{4} + \frac{1}{9} + \ \ldots\ + \frac{1}{k^2} + \ \ldots\ = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges.

We know that the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=2}^{\infty} \frac{1}{(k-1)k}$$

converges. For any $k = 2,\ 3,\ \ldots$ it is obvious that

$$\frac{1}{k^2} < \frac{1}{(k-1)k}$$

and by Theorem 2 the series

$$\sum_{k=2}^{\infty} \frac{1}{k^2}$$

converges. Adding the term 1 to the beginning of the series preserves the convergence.

Example 2. Prove that the series

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ \ldots\ +\frac{1}{\sqrt{k}}+\ \ldots\ =\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$$

diverges.

For any $k \geq 1$ there holds the inequality $\sqrt{k} \leq k$ hence,

$$\frac{1}{\sqrt{k}} > \frac{1}{k}$$

The harmonic series (8.3) diverges thus, by Theorem 2 the series given diverges also.

8.3.2 D'Alembert's test

Sometimes the D'Alembert's test is referred as the ratio test. We consider again the positive series (8.1).

Theorem (D'Alembert's test). Suppose there exists the limit

$$\lim_{k\to\infty}\frac{u_{k+1}}{u_k}=D$$

1) If $D < 1$, then the series (8.1) converges.

2) If $D > 1$, then series (8.1) diverges.

3) If $D = 1$, then this test us inconclusive, because there exist both convergent and divergent series that satisfy this case.

Proof. Suppose the limit $D < 1$ and let q be a real number between D and 1, i.e $D < q < 1$. By definition of the limit there exists $N > 0$ such that for $k \geq N$

$$\left|\frac{u_{k+1}}{u_k}-D\right| < q-D$$

which is equivalent to

$$-q+D < \frac{u_{k+1}}{u_k}-D < q-D$$

For $k \geq N$ the inequality on the right hand side gives

$$\frac{u_{k+1}}{u_k} < q$$

Thus, $u_{N+1} < qu_N$, $u_{N+2} < qu_{N+1} < q^2u_N$, ... Applying this $i-2$ more times, we get $u_{N+i} < q^iu_N$ and since $q < 1$, then

$$\sum_{i=0}^{\infty} q^iu_N < u_N\sum_{i=0}^{\infty} q^i = u_N\frac{1}{1-q}$$

Thus, the series

$$\sum_{i=0}^{\infty} q^i u_N$$

converges and by the comparison test

$$\sum_{i=0}^{\infty} u_{N+i} = \sum_{k=N}^{\infty} u_k$$

also converges. But then the series (8.1) is also convergent because adding $N-1$ of terms to the beginning preserves the convergence.

If $D > 1$, then for $1 < q < D$ by the definition of the limit there exists $N > 0$ such that for $k \geq N$

$$\left| \frac{u_{k+1}}{u_k} - D \right| < D - q$$

which yields

$$\frac{u_{k+1}}{u_k} - D > q - D$$

Hence,

$$\frac{u_{k+1}}{u_k} > q$$

or $u_{k+1} > qu_k > u_k$, i.e. the terms of the series form an increasing sequence. The limit of the increasing sequence cannot be zero thus, by divergence test the series (8.1) diverges.

Example 1. Does the series $\sum\limits_{k=1}^{\infty} \frac{1}{k!}$ converge or diverge?

The ratio of two consecutive terms $u_{k+1} = \frac{1}{(k+1)!}$ and $u_k = \frac{1}{k!}$ is

$$\frac{u_{k+1}}{u_k} = \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{k!}{(k+1)k!} = \frac{1}{k+1}$$

and the limit of this ratio

$$D = \lim_{k \to \infty} \frac{1}{k+1} = 0$$

Since $D = 0$, this series converges by the D'Alembert's test.

Example 2. Does the series $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ converge or diverge?

Compute the limit

$$D = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{\dfrac{1}{(k+1)^2}}{\dfrac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} = 1$$

Since $D = 1$, the D'Alembert's test is inconclusive, but we know that by the comparison test that this series converges.

Example 3. Does the series $\sum_{k=1}^{\infty} \frac{1}{k}$ converge or diverge?

For the harmonic series we have

$$D = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{\dfrac{1}{k+1}}{\dfrac{1}{k}} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

so, the harmonic series cannot be handled by the D'Alembert's test, but we know that the series diverges.

8.3.3 Cauchy test

Cauchy test is also known as root test of convergence of a series. Let us consider the positive series (8.1) again.

Theorem (Cauchy test). Suppose there exists the limit

$$\lim_{k\to\infty} \sqrt[k]{u_k} = C$$

1) If <1, then the series (8.1) converges.

2) If $C > 1$, then series (8.1) diverges.

3) If $C = 1$, then this test us inconclusive.

Proof. 1) Suppose the limit $C < 1$ and let q be a real number between C and 1, i.e $C < q < 1$. By definition of the limit there exists $N > 0$ such that for $k \geq N$

$$|\sqrt[k]{u_k} - C| < q - C$$

Hence, for $k \geq N$

$$-q + C < \sqrt[k]{u_k} - C < q - C$$

The inequality on the right hand side gives $\sqrt[k]{u_k} < q$ or $u_k < q^k$. Since $q < 1$, the geometric series

$$\sum_{k=N}^{\infty} q^k$$

converges hence, by the comparison test the series

$$\sum_{k=N}^{\infty} u_k$$

also converges. Adding $N-1$ terms to the beginning gives us the convergent series (8.1).

2) Suppose the limit $C > 1$ and let q be a real number between 1 and C, i.e $C > q > 1$. By definition of the limit there exists $N > 0$ such that for $k \geq N$

$$|\sqrt[k]{u_k} - C| < C - q$$

which is equivalent to

$$q - C < \sqrt[k]{u_k} - C < C - q$$

The inequality on the left hand side gives $\sqrt[k]{u_k} > q$ or $u_k > q^k$. Since $q > 1$, the limit

$$\lim_{k\to\infty} u_k$$

cannot equal to zero hence, by the divergence test the series (8.1) diverges.

Example 1. Determine if the series

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k}$$

is convergent or divergent?

To use the Cauchy test we find $\sqrt[k]{u_k} = \dfrac{\sqrt[k]{k^2}}{2}$ and evaluate the limit

$$C = \lim_{k\to\infty} \sqrt[k]{u_k} = \lim_{k\to\infty} \frac{\sqrt[k]{k^2}}{2} = \frac{1}{2} \lim_{k\to\infty} k^{\frac{2}{k}}$$

Since we have the indeterminate form ∞^0, we apply the L'Hospital's rule

$$\begin{aligned} \lim_{k\to\infty} \ln k^{\frac{2}{k}} &= \lim_{k\to\infty} \frac{2}{k} \ln k \\ = \lim_{k\to\infty} \frac{(2\ln k)'}{k'} &= \lim_{k\to\infty} \frac{2}{k} = 0 \end{aligned}$$

and

$$C = \frac{1}{2} e^0 = \frac{1}{2} < 1$$

So, by the Cauchy test the series is convergent.

Example 2. Determine if the series

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^{k^2}$$

is convergent or divergent?

The kth root of the general term is

$$\sqrt[k]{u_k} = \sqrt[k]{\left(1 + \frac{1}{k}\right)^{k^2}} = \left(1 + \frac{1}{k}\right)^k$$

and the limit

$$C = \lim_{k \to \infty} \sqrt[k]{u_k} = \lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^k = e > 1$$

Hence, by the Cauchy test the series is divergent.

8.3.4 Integral test

Let us consider the a positive series (8.1) once more.

Theorem 5 (Integral test). Suppose $u(x)$ is a continuous positive decreasing on interval $[1; \infty)$ function, whose values for the integer arguments are the terms of series (8.1), i.e. $u(k) = u_k$. Then

1) if the improper integral (8.1) $\int\limits_1^{\infty} u(x)dx$ is convergent so is the series (8.1);

1) if the improper integral (8.1) $\int\limits_1^{\infty} u(x)dx$ is divergent so is the series (8.1).

Proof. 1) By the assumption the improper integral

$$\int\limits_1^{\infty} u(x)dx$$

converges, i.e. there exists the finite limit

$$\lim_{N \to \infty} \int\limits_1^{N} u(x)dx$$

Stripping out the first term of the series (8.1), we obtain the series

$$\sum_{k=2}^{\infty} u_k$$

whose nth partial sum is

$$S_n = \sum_{k=2}^{N} u_k \tag{8.8}$$

The region in Figure 8.1 is bounded by the curve $u = u(x)$, x axis and the

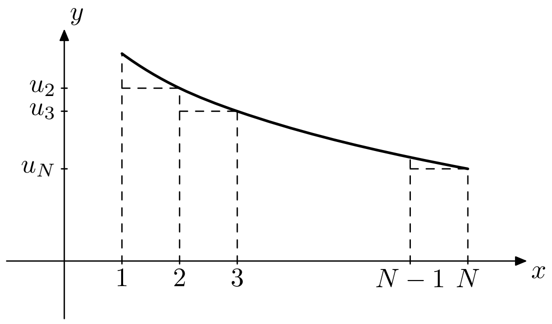


Figure 8.1.

lines $x = 1$ and $x = N$. We sketch in this region $N-1$ rectangles with bases 1 unit and heights u_2, u_3, ... , u_N, respectively. The sum of the areas of those rectangles is obviously less than the area under the graph of the function $u = u(x)$ if $x \geq 1$

$$S_n \leq \int_1^{\infty} u(x)dx$$

By the assumption the integral on the left side of this inequality is convergent, i.e. has the finite value. Hence, the sequence (8.8) is bounded and increasing since the terms of the series are positive. By Theorem 1 this sequence has the finite limit, i.e. the series

$$\sum_{k=2}^{\infty} u_k$$

converges. Consequently, the series

$$\sum_{k=1}^{\infty} u_k$$

is also convergent.

Now, suppose that the improper integral

$$\int_1^\infty u(x)dx$$

is divergent The sum of the areas of $N-1$ rectangles with bases 1

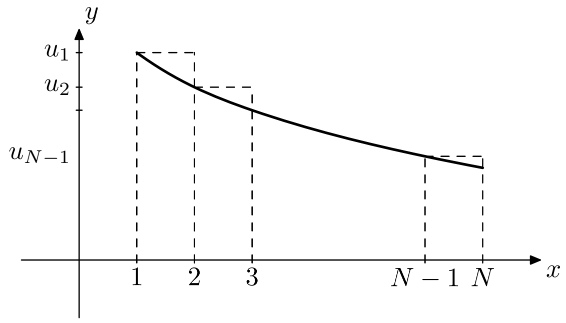


Figure 8.2.

$$u_1 \cdot 1 + u_2 \cdot 1 + ... + u_{N-1} \cdot 1 = S_{N-1}$$

is greater than the area under the graph

$$\int_1^N u(x)dx$$

and, since $u_N > 0$, then

$$\int_1^N u(x)dx \leq S_N$$

By the limit theorem

$$\lim_{N\to\infty} \int_1^N u(x)dx \leq \lim_{N\to\infty} S_N$$

The limit on the left side of this inequality is infinite hence, the limit on the right side has to be also infinite, i.e. the series

$$\sum_{k=1}^{\infty} u_k$$

diverges.

Example 4. Prove that the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges.

To apply the integral test we define the decreasing function $u(x) = \frac{1}{x}$, whose values for the integer arguments $x = k$ are

$$u_k = u(k) = \frac{1}{k}$$

The improper integral is divergent because

$$\int_1^{\infty} \frac{dx}{x} = \lim_{N \to \infty} \int_1^{N} \frac{dx}{x} = \lim_{N \to \infty} \ln|x| \Big|_1^N = \lim_{N \to \infty} \ln N = \infty$$

By the Integral test the harmonic series is divergent.

8.4 Alternating series. Leibnitz's test.

The last tests that we looked at for series convergence have required that all the terms in the series be positive. The test that we are going to look into in this subsection will be a test for alternating series. An alternating series is any series

$$u_1 - u_2 + u_3 - u_4 + \ \ldots = \sum_{k=1}^{\infty} (-1)^{k+1} u_k \tag{8.9}$$

or

$$-u_1 + u_2 - u_3 + u_4 - \ \ldots = \sum_{k=1}^{\infty} (-1)^{k} u_k$$

where $u_k > 0, \ \ k = 1, \ 2, \ \ldots$

The second alternating series we can write

$$\sum_{k=1}^{\infty} (-1)^{k} u_k = -\sum_{k=1}^{\infty} (-1)^{k+1} u_k$$

therefore, it's enough to look at for convergence of the series (8.9).

Theorem 1. (Leibnitz's test) If

1) $u_k > u_{k+1}$, $k = 1, \ 2, \ \ldots$ and

2) $\lim_{k\to\infty} u_k = 0$, then the alternating series (8.9) converges.

Proof. First, notice that because the terms of the series are decreasing for any two successive terms we have

$$u_k - u_{k+1} > 0$$

We will prove that both the partial sums S_{2n} with even indexes and S_{2n+1} with odd indexes converge to the same number S. First, consider the even partial sums

$$S_{2n} = \sum_{k=1}^{2n} (-1)^{k+1} u_k$$

We can write this partial sum as

$$S_{2n} = (u_1 - u_2) + (u_3 - u_4) + \ldots + (u_{2n-1} - u_{2n})$$

Each of the quantities in parenthesis are positive and one more pair $u_{2n+1} - u_{2n+2}$ increases the sum hence, the sequence of even partial sums is increasing. Next, we can also write this even partial sum as

$$S_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots - u_{2n}$$

Each of the quantities in parenthesis are positive again and u_{2n} is also positive. This gives us that

$$S_{2n} < u_1$$

for all n, i.e. the sequence of even partial sums is bounded. We now know that the increasing sequence that is bounded has the finite limit

$$\lim_{n\to\infty} S_{2n} = S$$

The nth term of the sequence of odd partial sums we write as

$$S_{2n+1} = S_{2n} + u_{2n+1}$$

and the second assumption of the theorem gives

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n+1} = S$$

Therefore,

$$\lim_{n\to\infty} S_n = S$$

which means that the series (8.9) converges.

Example. For the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$

both of the assumptions of the theorem hold because

$$1 > \frac{1}{2} > \ldots > \frac{1}{k} > \frac{1}{k+1} > \ldots$$

and

$$\lim_{k \to \infty} \frac{1}{k} = 0$$

Hence, this series is convergent.

8.5 Absolute and conditional convergence

In this subsection we assume that the terms of the series

$$\sum_{k=1}^{\infty} u_k \tag{8.10}$$

can have whatever signs.

Definition 1. The series (8.10) is called *absolutely convergent* if the series

$$|u_1| + |u_2| + |u_3| + \ldots = \sum_{k=1}^{\infty} |u_k|$$

is convergent.

Theorem 1. If the series (8.10) is absolutely convergent then it is also convergent.

Proof. The definition of the absolute value

$$|u_k| = \begin{cases} u_k, & \text{if } u_k \geq 0 \\ -u_k, & \text{if } u_k < 0 \end{cases}$$

gives us that

$$0 \leq u_k + |u_k| \leq 2|u_k|$$

Since we are assuming that

$$\sum_{k=1}^{\infty} |u_k|$$

is convergent then

$$\sum_{k=1}^{\infty} 2|u_k| = 2\sum_{k=1}^{\infty} |u_k|$$

is also convergent because 2 times finite value will still be finite. The comparison test gives us that

$$\sum_{k=1}^{\infty} (u_k + |u_k|)$$

is also a convergent series. Now the series (8.10)

$$\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} (u_k + |u_k| - |u_k|) = \sum_{k=1}^{\infty} (u_k + |u_k|) - \sum_{k=1}^{\infty} |u_k|$$

is the difference of two convergent series, i.e. convergent.

By Theorem 1 series that are absolutely convergent are guaranteed to be convergent. However, series that are convergent may or may not be absolutely convergent.

Definition 2. The series (8.10) which is convergent but not absolutely convergent is called *conditionally convergent.*

Example 1. Alternating harmonic series

$$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k}$$

is convergent by Leibnitz's test, but the series

$$\sum_{k=1}^{\infty} \left|(-1)^{k+1}\frac{1}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$$

is the harmonic series. By Integral test the harmonic series diverges hence, alternating harmonic series is a conditionally convergent series.

Example 2. Determine if the series $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ is absolutely convergent, conditionally convergent or divergent.

Notice that this is not an alternating series. Since $|\sin k| \leq 1$ for any integer k, then

$$\left|\frac{\sin k}{k^2}\right| = \frac{1}{k^2}$$

We know that the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges hence, by Comparison test the series

$$\left|\frac{\sin k}{k^2}\right|$$

converges, i.e. the series $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ is absolutely convergent and Theorem 1 guarantees its convergence.

While the convergence of the positive series takes place because of the terms are decreasing with the sufficient speed, then the conditional convergence happens because the terms reduce each other.

8.6 Series of functions

A series of functions is the series, whose terms are the functions of some variable

$$\sum_{k=1}^{\infty} u_k(x) \tag{8.11}$$

If we assign to the variable x a certain value x_0 that is in domains of all u_k and substitute it into all these functions, we have the numerical values $u_k(x_0)$, i.e for $x = x_0$ the series (8.11) is a number series.

Example. Let's examine the series of functions

$$1 + x + x^2 + \ldots + x^k + \ldots = \sum_{k=0}^{\infty} x^k \tag{8.12}$$

If the variable x has the value $x = \frac{1}{2}$, we get the geometric series

$$\sum_{k=0}^{\infty} \frac{1}{2^k}$$

which is convergent, because the common ratio is $\frac{1}{2}$.

Assigning to the variable x the value $x = 1$, we get the number series

$$1 + 1 + 1 + \ldots$$

which diverges by Divergence test. Assigning to the variable x the value $x = -1$, we get the divergent number series

$$1 - 1 + 1 - \ldots + (-1)^k + \ldots$$

Assigning to the variable x the some value $x_0 > 1$, we obtain the number series with general term

$$u_k(x_0) = x_0^k$$

which diverges by Divergence test because

$$\lim_{k\to\infty} x_0^k = \infty$$

Assigning to the variable x the some value $x_0 < -1$, we obtain the number series which diverges by Divergence test because the general term has no limit.

It has turned out that for some values of the variable x the series of functions converges and for other values it diverges.

The partial sums of the series of functions (8.11)

$$S_n(x) = \sum_{k=1}^{n} u_k(x)$$

are also functions of the variable x and define a sequence of functions

$$S_1(x),\ S_2(x),\ \ldots,\ S_n(x),\ \ldots \tag{8.13}$$

Definition. The set X of the values of argument x for which the sequence of partial sums (8.13) is convergent, i.e. there exists the (finite) limit

$$S(x) = \lim_{n\to\infty} S_n(x) \tag{8.14}$$

is called the *region of convergence* of the series of functions (8.11).

It is said that $S(x)$ is the sum of the series (8.11) and one writes

$$S(x) = \sum_{k=1}^{\infty} u_k(x)$$

The last equality can be written also as

$$S(x) = \sum_{k=1}^{n} u_k(x) + \sum_{k=n+1}^{\infty} u_k(x)$$

The term

$$R_n(x) = \sum_{k=n+1}^{\infty} u_k(x)$$

in this sum is called the remainder of the series of functions and

$$S(x) = S_n(x) + R_n(x)$$

8.7 Majorized series

Definition. The series of functions is said to be (8.11) *majorized* on a set X if there exists the positive convergent number series

$$\sum_{k=1}^{\infty} \alpha_k \tag{8.15}$$

such that for all $x \in X$ holds

$$|u_k(x)| \leq \alpha_k$$

The series (8.15) is called the *majorant series.*

Example. The series of functions

$$\sum_{k=1}^{\infty} \frac{\sin kx}{1+k^2}$$

is majorized on the set of real numbers $\mathbb{R}$ because for each $x \in \mathbb{R}$ there holds

$$\left|\frac{\sin kx}{1+k^2}\right| \leq \frac{1}{1+k^2}$$

and, since

$$\frac{1}{1+k^2} < \frac{1}{k^2}$$

the series

$$\sum_{k=1}^{\infty} \frac{1}{1+k^2}$$

converges by Comparison test

Theorem 1. If the series of functions (8.11) is to be majorized on the set X, then for every for all $x \in X$ the limit of the remainder

$$\lim_{n\to\infty} R_n(x) = 0$$

Proof. The majorant series (8.15) is a positive convergent number series. Denote the partial sums of this series

$$\sigma_n = \sum_{k=1}^{n} \alpha_k$$

and the sum

$$\sigma = \lim_{n\to\infty} \sigma_n$$

According to the definition of limit for all $\varepsilon > 0$ there exists $N > 0$ such that

$$|\sigma - \sigma_n| < \varepsilon$$

whenever $n \geq N$.

But

$$|\sigma - \sigma_n| = \left| \sum_{k=1}^{\infty} \alpha_k - \sum_{k=1}^{n} \alpha_k \right| = \left| \sum_{k=n+1}^{\infty} \alpha_k \right|$$

Denoting by

$$r_n = \sum_{k=n+1}^{\infty} \alpha_k$$

the remainder of the majorant series, we obtain that for all $\varepsilon > 0$ there exists $N > 0$ such that

$$r_n < \varepsilon$$

whenever $n \geq N$.

Since (8.11) is to be majorized by (8.15), then $|u_k(x)| < \alpha_k$ $(k = 1, 2, 3, \ldots$ $)$for all $x \in X$ thus,

$$|R_n(x)| = \left| \sum_{k=n+1}^{\infty} u_k(x) \right| \leq \sum_{k=n+1}^{\infty} |u_k(x)| < \sum_{k=n+1}^{\infty} \alpha_k = r_n < \varepsilon$$

i.e. for all $x \in X$

$$|R_n(x)| < \varepsilon$$

whenever $n > N$, which is we wanted to prove.

The last condition is equivalent to

$$|S(x) - S_n(x)| < \varepsilon \tag{8.16}$$

Definition. If for every $\varepsilon > 0$ there exists index $N > 0$ such that for all $n > N$ and for all $x \in X$ holds the condition (8.16), then the series of functions (8.11) is called uniformly convergent to $S(x)$ on the set X.

Conclusion. If the series of functions (8.11) is to be majorized on the set X, then it is uniformly convergent on X.

In following subsections we will see that the properties of finite sums of functions does not hold for the series of functions. But if we assume that the series of functions is uniformly convergent on the set X, then properties of finite sums of functions are still valid on that set.

As a rule it's easier to prove for the series of functions to be majorized on a set than to prove the uniformly convergence on that set. If we have succeeded to prove for the series of functions to be majorized on X, then Theorem 1 guarantees the uniform convergence of this series of functions on that set.

8.8 Continuity of sum of series of functions

One of the most important property of uniform convergence is that it preserves continuity.

The finite sum of continuous functions is a continuous function. The sum of infinite number of continuous functions may not be continuous. Let's consider on the interval $[0;1]$ the series of functions

$$\sum_{k=1}^{\infty} x^{k-1}(1-x) = (1-x) + x(1-x) + x^2(1-x) + \ldots + x^{k-1}(1-x) + \ldots$$

If $x = 1$, then all the terms of this series equal to 0 and, of course, $S(x) = 0$. If $x < 1$, the by the formula of the sum of geometric series

$$S(x) = (1-x)+x(1-x)+x^2(1-x)+\ldots+x^{k-1}(1-x)+\ldots = (1-x)\frac{1}{1-x} = 1$$

We see that at the point $x = 1$ the sum of this series is not continuous because

$$\lim_{x \to 1-} S(x) = 1$$

but $S(1) = 0$ despite of all the terms of this series are continuous on the whole set of real numbers.

Theorem. If the series of functions (8.11) continuous on the set X is uniformly convergent on X, then the sum of the series is continuous on X as well.

Proof. Let $S(x)$ be the sum of the series of functions (8.11). Prove that the necessary and sufficient condition for continuity

$$\lim_{\Delta x \to 0} \Delta S = 0 \tag{8.17}$$

where

$$\Delta S = S(x + \Delta x) - S(x) = S_n(x + \Delta x) + R_n(x + \Delta x) - S_n(x) - R_n(x)$$

is satisfied. By the property of the absolute value

$$|\Delta S| \leq |S_n(x + \Delta x) - S_n(x)| + |R_n(x + \Delta x)| + |R_n(x)| \tag{8.18}$$

Since the series is uniformly convergent on X, then by Theorem 1 of of the previous subsection for each $\varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$ and for all $x, x + \Delta x \in X$

$$|R_n(x + \Delta x)| < \frac{\varepsilon}{3}$$

and

$$|R_n(x)| < \frac{\varepsilon}{3}$$

The nth partial sum of the series (8.11) is the sum of the finite number of continuous functions, which is continuous. By the necessary and sufficient condition of continuity for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $|\Delta x| < \delta$ holds

$$|\Delta S_n| = |S_n(x + \Delta x) - S_n(x)| < \frac{\varepsilon}{3}$$

Now, the condition (8.18) yields that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $|\Delta x| < \delta$ we have

$$|\Delta S| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

which implies the condition (8.17), i.e. the sum $S(x)$ is continuous on X.

8.9 Term by term integration and differentiation of series of functions

Term-by-term integration and differentiation, the ability to find the integral or derivative of a sum of functions by integrating each summand, works for a finite sum. For the series of functions there holds the theorem.

Theorem 1. Suppose the functions $u_k(x)$, for each $k = 1,\ 2,\ \ldots$, are continuous on $[a;b]$ and the series (8.11) is uniformly convergent on $[a;b]$. Then the sum $S(x) = \sum_{k=1}^{\infty} u_k(x)$ is integrable on $[a;b]$ and

$$\int\limits_a^b S(x)dx = \sum_{k=1}^{\infty} \int\limits_a^b u_k(x)dx$$

i.e. the series (8.11) can be integrated term by term on $[a;b]$.

Proof. By the theorem of the previous subsection the sum

$$S(x) = \sum_{k=1}^{\infty} u_k(x)$$

is continuous on $[a;b]$ hence, there exists

$$\int\limits_a^b S(x)dx$$

To prove the theorem we have to show that the sequence of partial sums

$$\sum_{k=1}^{n} \int\limits_a^b u_k(x)dx$$

converges to $\int\limits_a^b S(x)dx$ i.e.

$$\int\limits_a^b S(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \int\limits_a^b u_k(x)dx$$

We estimate the difference

$$\left| \int\limits_a^b S(x)dx - \sum_{k=1}^{n} \int\limits_a^b u_k(x)dx \right| = \left| \int\limits_a^b \left[S(x) - \sum_{k=1}^{n} u_k(x) \right] dx \right| \leq$$

$$\leq \int\limits_a^b \left| S(x) - \sum_{k=1}^{n} u_k(x) \right| dx = \int\limits_a^b |R_n(x)|\, dx$$

Since the series of functions (8.11) converges uniformly, then for each $\varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$ and for all $x \in [a; b]$

$$|R_n(x)| < \frac{\varepsilon}{b-a}$$

Thus,

$$\int\limits_a^b |R_n(x)|\, dx < \frac{\varepsilon}{b-a} \int\limits_a^b dx = \varepsilon$$

which is we wanted to prove.

Conclusion 2. If $a \leq x_0 < x \leq b$ and for the series of functions (8.11) hold the assumptions of Theorem 1, then

$$\int\limits_{x_0}^{x} S(x)dx = \sum_{k=1}^{\infty} \int\limits_{x_0}^{x} u_k(x)$$

The conclusion is obvious because the functions $u_k(x)$ $(k = 1,\ 2,\ ...)$ are continuous on $[a; b]$ hence, also on $[x_0; x]$. Since the series (8.11) is uniformly convergent on $[a; b]$, it is also uniformly convergent on $[x_0; x]$.

Theorem 2. If the series (8.11) converges to $S(x)$ on $[a;b]$ and the series of derivatives

$$\sum_{k=1}^{\infty} u_k'(x)$$

converges uniformly to the sum $\sigma(x)$ on $[a;b]$, then $S'(x) = \sigma(x)$ or

$$\left[\sum_{k=1}^{\infty} u_k(x)\right]' = \sum_{k=1}^{\infty} u_k'(x)$$

i.e. the series (8.11) can be differentiated term by term.

Proof. By Conclusion 2 we get for $a \le x \le b$

$$\int_a^x \sigma(x)dx = \sum_{k=1}^{\infty} \int_a^x u_k'(x)dx = \sum_{k=1}^{\infty}(u_k(x) - u_k(a)) = \sum_{k=1}^{\infty} u_k(x) - \sum_{k=1}^{\infty} u_k(a)$$

or

$$\int_a^x \sigma(x)dx = S(x) - S(a)$$

and differentiating both sides of this equality with respect to x gives $\sigma(x) = S'(x)$.

8.10 Power series

Power series is a series of power functions

$$\sum_{k=0}^{\infty} c_k x^k \tag{8.19}$$

or in general

$$\sum_{k=0}^{\infty} c_k (x-a)^k \tag{8.20}$$

where the numbers c_k are called the coefficients of the series.

The examination of the properties of those series is very similar therefore, we restrict ourselves with series (8.19).

Example 1. The series

$$1 + x + x^2 + \ \ldots \ + x^k + \ \ldots = \sum_{k=0}^{\infty} x^k$$

is a geometric series for any value of x. This series converges if $|x| < 1$. Hence, the region of convergence of this series is open interval $X = (-1; 1)$ and the sum of this series in this interval is

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \tag{8.21}$$

It turns out that the regions of convergence of power series have such a simple structure.

Theorem 1 (Abel's theorem). If the power series (8.19) converges for some value of x_0, then this series converges absolutely for any value of $|x| < |x_0|$.

Conversely, if the power series (8.19) diverges for some value of x_0, then this series diverges for any value of $|x| > |x_0|$.

Proof. By the assumption, the number series

$$\sum_{k=0}^{\infty} c_k x_0^k$$

converges hence, the limit of the general term

$$\lim_{k\to\infty} c_k x_0^k = 0$$

The convergent sequence is bounded, i.e. there exists a constant $K > 0$ such that

$$|c_k x_0^k| < K$$

Let's denote $q = \dfrac{x}{x_0}$. Since $|x| < |x_0|$, then $|q| < 1$ and

$$|c_k x^k| = |c_k x_0^k \cdot \frac{x^k}{x_0^k}| < Kq^k$$

The geometric series

$$\sum_{k=0}^{\infty} Kq^k$$

is convergent thus, by Comparison test the series

$$\sum_{k=0}^{\infty} |c_k x^k|$$

is also convergent, i.e. the series (8.19) is absolutely convergent.

To prove another statement of this theorem assume on contrary that the series (8.19) converges for some $|x| > |x_0|$. But by the first part of the proof the series

$$\sum_{k=0}^{\infty} c_k x_0^k$$

should converge absolutely, which is contradictory to the assumption. Therefore, the series (8.19) cannot converge for $|x| > |x_0|$.

According to Abel's theorem there exists a real number R such that for $|x| < R$ the series (8.19) converges and for $|x| > R$ diverges. This real number R is called the *radius of convergence* of the series (8.19) and the interval $(-R; R)$ the *interval of convergence* of this series.

Remark. At the endpoints $x = R$ and $x = -R$ of the interval of convergence the series (8.19) may converge and may diverge. Therefore, to completely identify the interval of convergence all that we have to do is determine if the power series will converge for $x = R$ or $x = -R$. If the power series converges for one or both of these values then well need to include those in the interval of convergence.

There are a lot of possibilities to determine the radius of convergence of power series. One of these possibilities is given by the following theorem.

Theorem 2. Suppose that the coefficients of the series (8.19) c_k does not equal to 0 and there exists

$$d = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right|$$

then $R = d$.

Proof. Using Dalembert's test for convergence of the positive series

$$\sum_{k=1}^{\infty} |c_k x^k|$$

we get

$$D = \lim_{k\to\infty} \frac{|c_{k+1}x^{k+1}|}{|c_k x^k|} = |x| \lim_{k\to\infty} \frac{|c_{k+1}|}{|c_k|} = \frac{|x|}{d}$$

By Dalembert's test the series converges if $\frac{|x|}{d} < 1$, i.e. st $|x| < d$, and diverges if $\frac{|x|}{d} > 1$, i.e. $|x| > d$. Consequently the radius of convergence $R = d$ or

$$R = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right| \tag{8.22}$$

Example. Find the intervals of convergence of power series

$$\sum_{k=1}^{\infty} x^k$$

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$

and

$$\sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

The radius of convergence is 1 for all of three series. The coefficient of the first series are $c_k = 1$ hence,

$$R = \lim_{k\to\infty} \frac{1}{1} = 1$$

The coefficients of the second series are $c_k = \dfrac{1}{k}$ and

$$R = \lim_{k\to\infty} \frac{k+1}{k} = 1$$

The coefficients of the third series are $c_k = \dfrac{1}{k^2}$ and

$$R = \lim_{k\to\infty} \frac{(k+1)^2}{k^2} = 1$$

thus, all three series are convergent if $-1 < x < 1$ and diverges if $|x| > 1$. Determine if these series will converge for $x = 1$ or $x = -1$.

The general term of the first series at the right endpoint is $1^k = 1$, whose limit $1 \neq 0$ hence, the series diverges. At the left endpoint the general term is $(-1)^k$, which has no limit as $k \to \infty$, i.e. the series diverges again and the interval of convergence of the first series is $(-1; 1)$

The general term of the second series at the right endpoint is $\dfrac{1}{k}$ hence, the second series is at the right endpoint the harmonic series, which is divergent. At the left endpoint the general term is $\dfrac{(-1)^k}{k}$, i.e. the second series is at the left endpoint the alternating harmonic series, which converges by Leibnitz's test. Thus, the interval of convergence of the second series is $[-1; 1)$.

The general term of the second series at the right endpoint is $\frac{1}{k^2}$ and at the left endpoint $\frac{(-1)^k}{k^2}$. The absolute value of both of these is $\frac{1}{k^2}$. By Example 1 of subsection 8.3 the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges thus, the third series converges at both endpoints and the interval of convergence is $[-1; 1]$.

8.11 Uniform convergence of power series

Theorem. If the radius of convergence of the power series (8.19) is R, then this series is uniformly convergent on any interval $[a; b] \subset (-R; R)$.

Proof. Suppose $r = \max\{|a|, |b|\}$. Then

$$[a; b] \subset [-r; r] \subset (-R; R)$$

Since r is an interior point of the interval of convergence, then

$$\sum_{k=1}^{\infty} c_k r^k$$

is absolutely convergent, i.e.

$$\sum_{k=1}^{\infty} |c_k| r^k$$

is a positive convergent number series. For any $x \in [a; b]$

$$|c_k x^k| \leq |c_k| r^k$$

i.e. the series (8.19) is to be majorized on $[a; b]$ hence, by Conclusion of the subsection 8.7 it is also uniformly convergent on $[a; b]$.

Now it's possible to have three conclusions.

Conclusion 1. If the radius of convergence of the power series (8.19) is R, then the sum of this series is continuous on any interval $[a; b] \subset (-R; R)$.

Since there holds the Theorem of subsection 8.8, this is obvious.

Conclusion 2. If the radius of convergence of the power series (8.19) is R, then this series can be integrated term by term on any interval $[a; b] \subset (-R; R)$.

Due to the Theorem 1 of subsection 8.9, this is obvious again.

Conclusion 3. If the radius of convergence of the power series (8.19) is R, then this series can be differentiated term by term on any interval $[a; b] \subset (-R; R)$.

Proof. If we differentiate the power series (8.19) term by term, we get

$$\sum_{k=1}^{\infty} k \cdot c_k x^{k-1}$$

According to the Theorem 2 of subsection 8.9 we have to show that the radius of convergence of the series obtained is still R. We find the radius of convergence of this series R' by the formula (8.22)

$$R' = \lim_{k\to\infty} \left| \frac{kc_k}{(k+1)c_{k+1}} \right| = \lim_{k\to\infty} \frac{k}{k+1} \cdot \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right| = 1 \cdot R$$

which is we wanted to prove.

Now, using the sum of the geometric series (8.21) and conclusions 2 and 3, we can find the power series expansions for many functions.

Example 1. Multiplying both sides of (8.21) by x gives

$$\frac{x}{1-x} = x \cdot \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} x^{k+1}$$

and the radius of convergence is still 1. It's easy to verify that

$$\left(\frac{x}{1-x} \right)' = \frac{1}{(1-x)^2}$$

and using the term by term differentiation we get the power series expansion of this derivative

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (x^{k+1})' = \sum_{k=0}^{\infty} (k+1)x^k$$

and the radius of convergence of the series obtained is 1 again.

Example 2. If we substitute in (8.21) the variable x by $-x^2$, we get

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

and this series converges if $|-x^2| < 1$, which is equivalent to $|x| < 1$.

Since

$$\arctan x = \int_0^x \frac{dx}{1+x^2}$$

, we obtain the power series of arc tangent function integrating the last series term by term in limits from 0 to x provided $|x| < 1$.

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \int_0^x x^{2k} dx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

and the radius of convergence is 1 hence, the interval of convergence is $(-1; 1)$.

At the left endpoint of the interval of convergence we get the series

$$\sum_{k=0}^{\infty} (-1)^k \frac{(-1)^{2k+1}}{2k+1} = -\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

and at the right endpoint

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

Both series obtained are the alternating series, which converge by the Leibnitz's test and therefore, the interval of convergence of the series obtained is $[-1; 1]$.

So, it may happen that the series obtained as the result of term by term integration converges at one or both of the endpoints, despite of the initial series diverges at the endpoints.

8.12 Taylor's and Maclaurin's series

Suppose that the function $f(x)$ is differentiable infinitely many times in the neighborhood of a. If the coefficients c_k of the power series

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

are computed by the formula

$$c_k = \frac{f^{(k)}(a)}{k!} \tag{8.23}$$

then these coefficients are called Taylor's coefficients and the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k \tag{8.24}$$

is called Taylor's series of the function $f(x)$ in the neighborhood of a or Taylor's series of the function $f(x)$ in powers $x-a$. The nth partial sum of this series is the Taylor's polynomial

$$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

By Taylor's formula the function $f(x)$ can be represented as

$$f(x)=P_n(x)+R_n(x)$$

that is the sum of the Taylor's polynomial and the remainder.

We know that Lagrange form of the remainder of the Taylor's formula is

$$R_n(x)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(a+\Theta(x-a))$$

where $0<\Theta<1$

If

$$\lim_{n\to\infty}R_n(x)=0$$

then

$$\lim_{n\to\infty}P_n(x)=f(x)$$

which means that the sequence of partial sums of Taylor's series converges to the function $f(x)$.

Therefore, the series (8.24) represents the function $f(x)$ if and only if the limit of the remainder equals to 0. If $\lim\limits_{n\to\infty}R_n(x)\neq 0$, then the Taylor's series of the function $f(x)$ may still converge but it does not represent the function $f(x)$.

Taylor's series in the neighborhood of $a=0$, i.e. Taylor's series in powers x

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k \tag{8.25}$$

is called Maclaurin's series.

8.13 Maclaurin's series of functions e^x, $\sin x$ and $\cos x$

In Mathematical analysis I we have proved that Maclaurin's formula of nth degree of the exponential function e^x is

$$e^x=1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\ldots+\frac{1}{n!}x^n+R_n(x)$$

and that the limit of the remainder

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} e^{\Theta x} = 0$$

for each $x \in \mathbb{R}$ and for $0 < \theta < 1$. Consequently, Maclaurin's series represents the function e^x for every real x, i.e.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Also it has been proved that Maclaurin's formula of $2n+1$st degree of the sine function $\sin x$ is

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n+1}(x)$$

whose remainder is

$$R_{2n+1}(x) = \frac{x^{2n+2}}{(2n+2)!} \sin\left(\Theta x + (n+1)\pi\right)$$

Since for every $x \in \mathbb{R}$ and for $0 < \theta < 1$

$$\lim_{n\to\infty} R_{2n+1}(x) = 0$$

Maclaurin's series represents the function $\sin x$ for every real x:

$$\sin x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

As well it has been proved that Maclaurin's formula of $2n$th degree of the cosine function $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n}(x)$$

and the remainder

$$R_{2n}(x) = \frac{x^{2n+1}}{(2n+1)!} \cos\left(\Theta x + (2n+1)\frac{\pi}{2}\right)$$

Again, for every $x \in \mathbb{R}$ and for $0 < \theta < 1$

$$\lim_{n\to\infty} R_{2n}(x) = 0$$

hence, Maclaurin's series represents the function $\cos x$ for every real x:

$$\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$

8.14 Trigonometric system of functions

The system of functions

$$\{1;\ \sin x;\ \cos x;\ \sin 2x;\ \cos 2x;\ \ldots;\ \sin kx;\ \cos kx;\ \ldots\} \tag{8.26}$$

is called the *trigonometric system* of functions. Let's find the definite integrals in limits from $-\pi$ to π of the products of two functions of the trigonometric system. To find these integrals we use three formulas of trigonometry

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \tag{8.27}$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)] \tag{8.28}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \tag{8.29}$$

First we find the definite integral of products of two sine functions $\sin kx$ and $\sin nx$ ($k = 1,\ 2,\ \ldots,\ n = 1,\ 2,\ \ldots$). By the formula (8.29)

$$\int\limits_{-\pi}^{\pi} \sin kx \sin nx dx = \frac{1}{2}\int\limits_{-\pi}^{\pi}[\cos(k-n)x - \cos(k+n)x]dx$$

If $n \neq k$, then

$$\int\limits_{-\pi}^{\pi} \sin kx \sin nx dx = \frac{1}{2(k-n)}\sin(k-n)x\Big|_{-\pi}^{\pi} - \frac{1}{2(k+n)}\sin(k+n)x\Big|_{-\pi}^{\pi} = 0$$

If $n = k$, then

$$\int\limits_{-\pi}^{\pi} \sin kx \sin kx dx = \frac{1}{2}\int\limits_{-\pi}^{\pi}(1-\cos 2kx)dx = \frac{1}{2}x\Big|_{-\pi}^{\pi} - \frac{1}{4k}\sin 2kx\Big|_{-\pi}^{\pi} = \pi$$

The first function of the trigonometric system can be considered as $1 = \cos 0x$. Next we find the definite integral of products of two cosine functions $\cos kx$ and $\cos nx$ ($k = 0,\ 1,\ 2,\ \ldots,\ n = 0,\ 1,\ 2,\ \ldots$). By the formula (8.28)

$$\int\limits_{-\pi}^{\pi} \cos kx \cos nx dx = \frac{1}{2}\int\limits_{-\pi}^{\pi}[\cos(k+n)x + \cos(k-n)x]dx$$

If $n \neq k$, then

$$\int_{-\pi}^{\pi} \cos kx \cos nx dx = \frac{1}{2(k+n)} \sin(k+n)x \Big|_{-\pi}^{\pi} + \frac{1}{2(k-n)} \sin(k-n)x \Big|_{-\pi}^{\pi} = 0$$

If $n = k \neq 0$, then

$$\int_{-\pi}^{\pi} \cos kx \cos kx dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos 2kx + 1) dx = \frac{1}{4k} \sin 2kx \Big|_{-\pi}^{\pi} + \frac{1}{2} x \Big|_{-\pi}^{\pi} = \pi$$

If $n = k = 0$, then

$$\int_{-\pi}^{\pi} \cos kx \cos nx dx = \frac{1}{2} \int_{-\pi}^{\pi} (1+1) dx = 2\pi$$

Third we find the definite integrals of the products of $\sin kx$ $(k = 1,\ 2,\ \ldots)$ and $\cos nx$ $(n = 0,\ 1,\ 2,\ \ldots)$. By the formula (8.27)

$$\int_{-\pi}^{\pi} \sin kx \cos nx dx = \frac{1}{2} \int_{-\pi}^{\pi} [\sin(k+n)x + \sin(k-n)x] dx$$

If $n \neq k$, then

$$\int_{-\pi}^{\pi} \sin kx \cos nx dx = -\frac{1}{2(k+n)} \cos(k+n)x \Big|_{-\pi}^{\pi} - \frac{1}{2(k-n)} \cos(k-n)x \Big|_{-\pi}^{\pi} = 0$$

If $n = k$, then

$$\int_{-\pi}^{\pi} \sin kx \cos kx dx = \frac{1}{k} \int_{-\pi}^{\pi} \sin kx d(\sin kx) = \frac{1}{k} \cdot \frac{\sin^2 kx}{2} \Big|_{-\pi}^{\pi} = 0$$

Consequently, the definite integrals over the interval $[-\pi; \pi]$ of the products of two different functions of trigonometric system always equal to 0. The definite integrals over the interval $[-\pi; \pi]$ of the squares of functions of the trigonometric system equal to π. The only exception is the first function 1, whose square integrated over $[-\pi; \pi]$ gives 2π. The summary of those computations is:

$$\int_{-\pi}^{\pi} \sin kx \sin nx dx = \begin{cases} 0 & \text{if} \quad n \neq k \\ \pi & \text{if} \quad n = k \neq 0 \end{cases} \tag{8.30}$$

$$\int_{-\pi}^{\pi} \cos kx \cos nx dx = \begin{cases} 0 & \text{if} \quad n \neq k \\ \pi & \text{if} \quad n = k \neq 0 \\ 2\pi & \text{if} \quad n = k = 0 \end{cases} \tag{8.31}$$

$$\int_{-\pi}^{\pi} \sin kx \cos nx dx = 0 \text{ for all } k \text{ and } n \tag{8.32}$$

It is said, that the trigonometric system is the *orthogonal system of functions* on the interval $[-\pi; \pi]$.

8.15 Fourier series of 2π-periodic functions

Recall that the function $f(x)$ is 2π-periodic if for each $x, x + 2\pi \in X$

$$f(x + 2\pi) = f(x)$$

which means that the values of the function are repeated at interval 2π in its domain.

Suppose the 2π-periodic function $f(x)$ can be expanded into uniformly convergent trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{8.33}$$

We shall see later that taking the constant term as $\dfrac{a_0}{2}$ rather that a_0 is a convenience that enables us to make a_0 fit a general result.

The uniform convergence of (8.33) on the set of real numbers is guaranteed if the positive number series

$$\sum_{k=1}^{\infty} (|a_k| + |b_k|)$$

converges because for any real x

$$|a_k \cos kx + b_k \sin kx| \leq |a_k| + |b_k|$$

which means that the series (8.33) is to be majorized and therefore by Theorem 1 of subsection 8.7 uniformly convergent on the set of real numbers. By Theorem 1 of the subsection 8.9 the series (8.33) can be integrated term-by-term in limits from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} \frac{a_0}{2} dx + \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} (a_k \cos kx + b_k \sin kx) dx$$

Since for all $k = 1, 2, \dots$

$$\int_{-\pi}^{\pi} \cos kx = 0$$

and

$$\int_{-\pi}^{\pi} \sin kx = 0$$

then

$$\int_{-\pi}^{\pi} f(x)dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx = \frac{a_0}{2} \cdot 2\pi = a_0 \cdot \pi$$

hence,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \tag{8.34}$$

To find the coefficients a_k of the expansion (8.33) we multiply both sides of this equality by $\cos nx$, assuming $n \geq 1$:

$$f(x)\cos nx = \frac{a_0}{2}\cos nx + \sum_{k=1}^{\infty}(a_k \cos kx \cos nx + b_k \sin kx \cos nx)$$

The series obtained is still uniformly convergent since

$$|a_k \cos kx \cos nx + b_k \sin kx \cos nx| \leq |a_k \cos kx + b_k \sin kx|$$

and we can integrate this series term-by-term in limits from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{a_0}{2}\int_{-\pi}^{\pi} \cos nxdx + \sum_{k=1}^{\infty}(a_k \int_{-\pi}^{\pi} \cos kx \cos nxdx + b_k \int_{-\pi}^{\pi} \sin kx \cos nxdx)$$

The first integral on the right side of this equality equals to 0. The orthogonality conditions (8.31) and (8.32) give that the only summand different from 0 and equal to π in the infinite sum is the term $k = n$ thus,

$$\int_{-\pi}^{\pi} f(x)\cos nxdx = a_n \cdot \pi$$

which gives (if we substitute n by k)

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kxdx \quad k = 1, 2, \dots \tag{8.35}$$

Similarly, multiplying both sides of the expansion (8.33) by $\sin nx$, assuming $n \geq 1$, we get

$$b_k = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin kx dx \quad k = 1,\ 2,\ \ldots \tag{8.36}$$

The coefficients a_0, a_k and b_k defined by (8.34), (8.35) and (8.36), respectively, are called the *Fourier coefficients* of the function $f(x)$ and the trigonometric series with these coefficients is called the *Fourier series* of the function $f(x)$.

We have got the formulas to compute the Fourier coefficients, assuming that the series is uniformly convergent on the set of real numbers. But if we compute the Fourier coefficients by the formulas (8.34), (8.35) and (8.36) and write the Fourier series expansion

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

we don't know whether this expansion converges and if it converges, converges it to $f(x)$ or to some other value. For now we are just saying that associated with the function $f(x)$ on $[-\pi;\pi]$ is a certain series called Fourier series. Therefore we write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \tag{8.37}$$

The equality sign $=$ can be written instead of $\sim$ only if we have proved the convergence of the Fourier series to the function $f(x)$.

Example 1. Find the Fourier coefficients and Fourier series of the square-wave function defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad -\pi < x \leq 0 \\ 1 & \text{if} \quad 0 < x \leq \pi \end{cases} \qquad \text{and} \quad f(x + 2\pi) = f(x)$$

So $f(x)$ is periodic with period 2π and its graph is shown in Figure 8.3. Using the formulas (8.34), (8.35) and (8.36), we find the Fourier coefficients

$$a_0 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int\limits_{0}^{\pi} dx = \frac{1}{\pi}\cdot\pi = 1$$

$$a_k = \frac{1}{\pi}\int\limits_{0}^{\pi} \cos kx dx = \frac{1}{k\pi}\sin kx\bigg|_0^{\pi} = 0$$

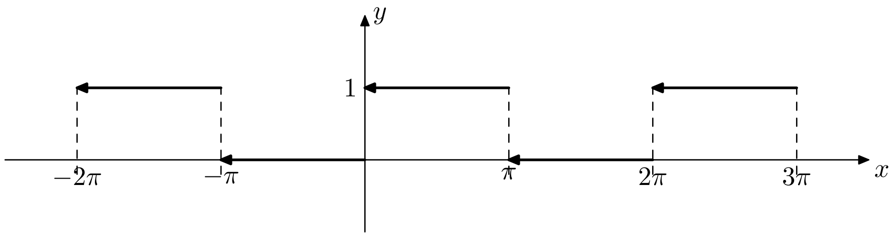


Figure 8.3.

and

$$b_k = \frac{1}{\pi}\int_0^{\pi} \sin kx dx = -\frac{1}{k\pi}\cos kx\Big|_0^{\pi} = -\frac{1}{k\pi}((-1)^k - 1) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{2}{k\pi} & \text{if } k \text{ is odd} \end{cases}$$

Thus, $a_k = 0$ and and $b_{2k} = 0$ for every $k = 1,\ 2,\ \ldots$. Fourier series of square-wave function is

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \ldots$$

or

$$f(x) \sim \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi}\sin(2k+1)x$$

The following theorem gives a sufficient condition for convergence of the Fourier series.

Theorem (Dirichlet's theorem). If $f(x)$ is a bounded 2π-periodic function which in any one period has at most a finite number of local maxima and minima and a finite number of points of jump discontinuity, then the Fourier series of $f(x)$ converges to $f(x)$ at all points where $f(x)$ is continuous and converges to the average of the right- and left-hand limits of $f(x)$ at each point where $f(x)$ is discontinuous.

The square-wave function has on half-open interval $(-\pi;\pi]$ one local maximum equal to 1 and one local minimum equal to 0 and two points of jump discontinuity 0 an π. Hence, at any point in the open intervals $(-\pi;0)$ and $(0;\pi)$ Fourier series converges to $f(x)$. The left-hand limit at 0 is $f(0-) = \lim\limits_{x\to 0-} f(x) = 0$ and the right-hand limit at 0 is $f(0+) = \lim\limits_{x\to 0+} f(x) = 1$ and the average of these one-sided limits is $\dfrac{0+1}{2} = \dfrac{1}{2}$. The left-hand limit at π is $f(\pi-) = \lim\limits_{x\to\pi-} f(x) = 1$ and the right-hand limit at π is $f(\pi+) = \lim\limits_{x\to\pi+} f(x) = 0$ and the average of one-sided limits is $\dfrac{1+0}{2} = \dfrac{1}{2}$.

Thus, at the points of discontinuity the Fourier series of the square-wave function converges to $\frac{1}{2}$. Since $\sin((2k+1)\cdot 0) = 0$ and $\sin((2k+1)\pi) = 0$ for any integer k, then the direct computation also gives

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin((2k+1)0) = \frac{1}{2}$$

and

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin((2k+1)\pi) = \frac{1}{2}$$

Figure 8.4 shows the graphs of the partial sums

$$S_{2n+1} = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \ldots + \frac{2}{(2n+1)\pi}\sin(2n+1)x$$

for $n = 0,\ 1,\ 2,\ 3$.

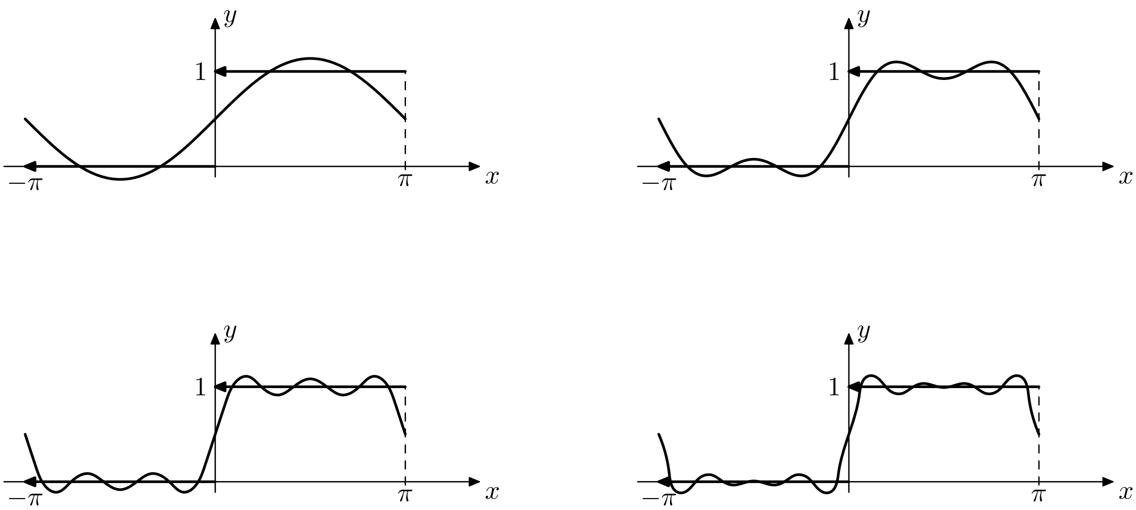


Figure 8.4. Partial sums of the Fourier series for the square-wave function

8.16 Fourier sine and cosine series of 2π-periodic functions

In some of the problems that we encounter, the Fourier coefficients become zero after integration. Finding zero coefficients in such problems can be avoided because using knowledge of even and odd functions, a zero coefficient may be predicted without performing the integration.

Corollary 1. If $f(x)$ is even, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

and if $f(x)$ is odd, then

$$\int_{-a}^{a} f(x)dx = 0$$

Proof. By the additivity property of the definite integral

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

and substituting in the first addend on the right side of this equality $x = -t$, we have $dx = -dt$. If $x = 0$ then $t = 0$ and if $x = -a$, then $t = a$. Thus,

$$\int_{-a}^{a} f(x)dx = -\int_{a}^{0} f(-t)dt + \int_{0}^{a} f(x)dx = \int_{0}^{a} f(-t)dt + \int_{0}^{a} f(x)dx$$

and, denoting in the first addend the variable of integration by x again

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} [f(-x) + f(x)]dx$$

Now, if $f(x)$ is even, then $f(-x) + f(x) = 2f(x)$ and if $f(x)$ is odd, then $f(-x) + f(x) = 0$.

Recall from the course of Mathematical analysis 1 three assertions about even and odd functions:

- the product of two even functions in an even function;
- the product of two odd functions is an even function;
- the product of an even and an odd function is an odd function.

Suppose that $f(x)$ is an even 2π-periodic function. Then for any integer $k \geq 1$ the product $f(x)\cos kx$ is even and the product $f(x)\sin kx$ is odd. Now, Corollary 1 simplifies the computation of Fourier coefficients:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cos kx dx, \quad k \geq 1$$

and

$$b_k = 0, \quad k \geq 1$$

Therefore, an even function $f(x)$ has only cosine terms in its Fourier expansion

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

This series is called a *Fourier cosine series.*

Next, suppose that $f(x)$ is an odd 2π-periodic function. Then for any integer $k \geq 1$ the product $f(x)\cos kx$ is odd and the product $f(x)\sin kx$ is even. Again, Corollary 1 simplifies the computation of Fourier coefficients:

$$a_0 = 0$$

$$a_k = 0, \quad k \geq 1$$

and

$$b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin kx dx, \quad k \geq 1$$

Thus, an odd function $f(x)$ has only sine terms in its Fourier expansion

$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin kx$$

and this series is called a *Fourier sine series.*

Example 1. Find the Fourier coefficients and Fourier series of the function defined by

$$f(x) = |x|, \text{ if} -\pi < x \leq \pi \quad \text{and} \quad f(x+2\pi) = f(x)$$

The graph of this 2π-periodic function is in Figure 8.5.

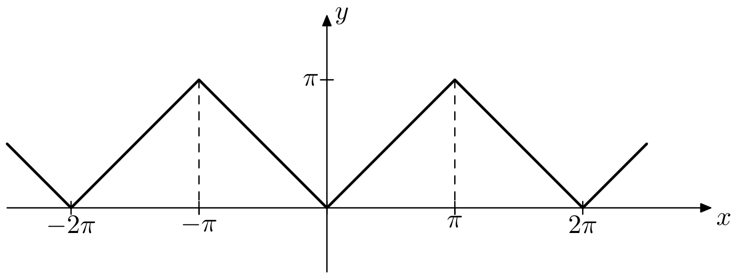


Figure 8.5.

The function is even and on the interval $[0; \pi]$ the absolute value $|x| = x$. We know, that the Fourier coefficients $b_k = 0$ for $k \geq 1$. Find

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \left.\frac{x^2}{2}\right|_0^{\pi} = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$$

and for $k \geq 1$ we integrate by parts

$$\begin{aligned} a_k &= \frac{2}{\pi}\int_0^{\pi} x\cos kx dx = \frac{2}{\pi}\left(\left.\frac{1}{k}x\sin kx\right|_0^{\pi} - \frac{1}{k}\int_0^{\pi}\sin kx dx\right) = \\ &= \left.\frac{2}{\pi k^2}\cos kx\right|_0^{\pi} = \frac{2\cdot(-1)^k}{\pi k^2} - \frac{2}{\pi k^2} = \begin{cases} -\dfrac{4}{\pi k^2}, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

The Fourier cosine series expansion of the function given

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x - \frac{4}{25\pi}\cos 5x - \ldots$$

or

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}$$

The function is continuous and has in the interval $(-\pi; \pi]$ one local maximum and one local minimum. Hence, by Dirichlet's convergence theorem the Fourier series converges to $f(x)$ at any real x and now we may write for all real x

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}$$

In particular

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2}$$

for $-\pi < x \le \pi$ Since $f(0) = 0$, then taking in the last equality $x = 0$, we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

or

$$\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi}{2}$$

which gives

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

Let's denote the sum of the convergent series

$$S = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Writing

$$\begin{aligned} S &= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \ldots = \\ &= 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \ldots + \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots\right) = \frac{\pi^2}{8} + \frac{1}{4} \cdot S \end{aligned}$$

gives

$$\frac{3}{4} \cdot S = \frac{\pi^2}{8}$$

or

$$S = \frac{\pi^2}{6}$$

Thus, the sum of the reciprocals of the squares of positive integers

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

The convergent Fourier series enable us to find a lot of sums of the kind.

Example 2. Find the Fourier coefficients and Fourier series of the rectangular wave defined by

$$f(x) = \begin{cases} -1 & \text{if} \quad -\pi < x < 0 \\ 1 & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \quad f(x + 2\pi) = f(x)$$

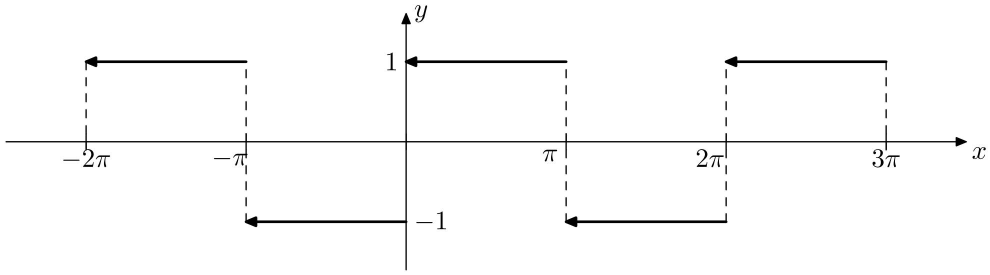


Figure 8.6.

The graph of this 2π-periodic function is in Figure 8.6.
The function is odd hence,

$$a_k = 0 \quad \text{for} \quad k = 0,\ 1,\ 2,\ \ldots$$

and

$$b_k = \frac{2}{\pi}\int_0^{\pi} \sin kx dx = -\frac{2}{k\pi}\cos kx\Big|_0^{\pi} = -\frac{2}{k\pi}((-1)^k - 1) = \begin{cases} \dfrac{4}{k\pi}, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even} \end{cases}$$

The Fourier sine series expansion of the function given

$$f(x) \sim \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \ldots$$

or

$$f(x) \sim \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}$$

By Dirichlet' convergent theorem for any $x \in ((2k-1)\pi; 2k\pi)$ the series converges to -1, for any $x \in (2k\pi; (2k+1)\pi)$ the series converges to 1 and at the points of jump discontinuity $x = k\pi$ the series converges to $\dfrac{-1+1}{2} = 0$.

8.17 Fourier series of functions with whatever period

If a function has period other than 2π, we can find its Fourier series by making a change of variable. Suppose $f(x)$ has period T, that is $f(x+T) = f(x)$ for all x. Then the function $f\left(\frac{T}{2\pi}\cdot x\right)$ has the period 2π because

$$f\left(\frac{T}{2\pi}(x+2\pi)\right) = f\left(\frac{Tx}{2\pi} + T\right) = f\left(\frac{Tx}{2\pi}\right)$$

The Fourier series of the function $f\left(\frac{Tx}{2\pi}\right)$ is

$$f\left(\frac{Tx}{2\pi}\right) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Tx}{2\pi}\right) dx$$

and, for $k = 1,\ 2,\ 3,\ \ldots$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Tx}{2\pi}\right) \cos kx dx$$

and

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Tx}{2\pi}\right) \sin kx dx$$

If we use the substitution $t = \dfrac{Tx}{2\pi}$, we have $x = \dfrac{2\pi t}{T}$ and $dx = \dfrac{2\pi}{T}dt$. If $x = -\pi$, then $t = -\dfrac{T}{2}$ and if $x = \pi$, then $t = \dfrac{T}{2}$. Therefore, the Fourier series of the function $f(t)$ with period T is

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{2k\pi t}{T} + b_k \sin\frac{2k\pi t}{T}$$

where

$$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) dt$$

and, for $k = 1,\ 2,\ 3,\ \ldots$

$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos\frac{2k\pi t}{T} dt$$

and

$$b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin\frac{2k\pi t}{T} dt$$

If T is the period of the function $f(x)$, then the ratio $\omega = \dfrac{2\pi}{T}$ is called the *angular frequency* or simply the *frequency* of the function $f(x)$. Denoting in the Fourier series expansion and in the formulas of Fourier coefficients the variable by x again, we have the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega x + b_k \sin k\omega x$$

where

$$a_0 = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx$$

and, for $k = 1,\ 2,\ 3,\ \ldots$

$$a_k = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos k\omega x dx \quad \text{and} \quad b_k = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \sin k\omega x dx$$

Of course, the Dirichlet' convergence theorem is also valid for functions with period T.

Example. Find the Fourier series expansion for

$$f(x) = \begin{cases} 0, & \text{if } \ -2 < x < 0 \\ x, & \text{if } \ 0 \leq x \leq 2 \end{cases} \quad \text{and} \quad f(x+4) = f(x)$$

The graph of this function is shown in Figure 8.7. The function has the

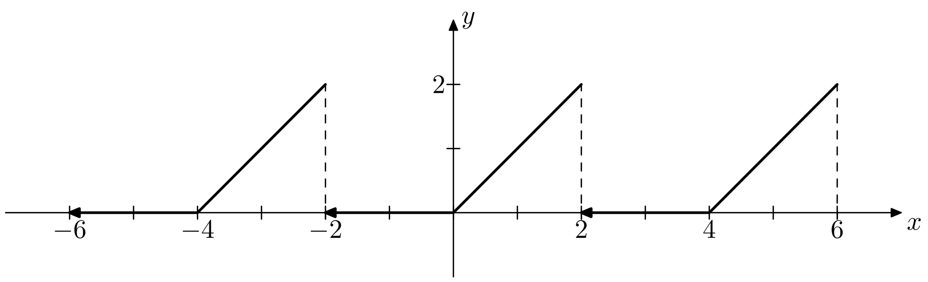


Figure 8.7.

period $T = 4$ and the frequency $\omega = \frac{2\pi}{4} = \frac{\pi}{2}$. Since the function equals to 0 between -2 and 0, then

$$a_0 = \frac{2}{4}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{-2}^{0} 0 \cdot dx + \frac{1}{2}\int_{0}^{2} xdx = \frac{1}{2} \cdot \frac{x^2}{2}\bigg|_0^2 = 1$$

Integration by parts gives

$$\begin{aligned} a_k &= \frac{1}{2}\int_0^2 x\cos\frac{k\pi x}{2}dx = \\ &= \frac{1}{2}\left[\frac{2}{k\pi}x\sin\frac{k\pi x}{2}\bigg|_0^2 + \frac{2^2}{k^2\pi^2}\cos\frac{k\pi x}{2}\bigg|_0^2\right] = \\ &= \frac{2}{k^2\pi^2}((-1)^k - 1) \end{aligned}$$

As well, integrating by parts, we get

$$\begin{aligned} b_k &= \frac{1}{2}\int_0^2 x\sin\frac{k\pi x}{2}dx = \\ &= \frac{1}{2}\left[-\frac{2}{k\pi}x\cos\frac{k\pi x}{2}\bigg|_0^2 + \frac{4}{k^2\pi^2}\sin\frac{k\pi x}{2}\bigg|_0^2\right] = \\ &= -\frac{2}{k\pi}(-1)^k = \frac{2 \cdot (-1)^{k+1}}{k\pi} \end{aligned}$$

So, the Fourier series expansion of the function is

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\left[\frac{(-1)^k - 1}{k^2\pi}\cos\frac{k\pi x}{2} + \frac{(-1)^{k+1}}{k}\sin\frac{k\pi x}{2}\right]$$

Noticing, that the numerators of the coefficients of cosine functions are 0 for even k-s and -2 for odd k-s, we may re-write this expansion as

$$f(x) \sim \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2\pi}\cos\frac{(2k+1)\pi x}{2} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sin\frac{k\pi x}{2}$$

8.18 Fourier series of half range functions

If the function with period T is even, then the Fourier coefficients for $k = 0,\ 1,\ 2,\ 3,\ \ldots$

$$a_k = \frac{4}{T}\int\limits_0^{\frac{T}{2}} f(x)\cos k\omega x dx \tag{8.38}$$

and for $k = 1,\ 2,\ 3,\ \ldots$

$$b_k = \frac{2}{T}\int\limits_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin k\omega x dx = 0 \tag{8.39}$$

The Fourier series expansion of the even function is

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega x \tag{8.40}$$

If the function with period T is odd, then for $k = 0,\ 1,\ 2,\ \ldots$ the Fourier coefficients

$$a_k = \frac{2}{T}\int\limits_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos k\omega x dx = 0 \tag{8.41}$$

and for $k = 1,\ 2,\ \ldots$

$$b_k = \frac{4}{T}\int\limits_0^{\frac{T}{2}} f(x)\sin k\omega x dx \tag{8.42}$$

and the Fourier series expansion of the odd function is

$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin k\omega x \tag{8.43}$$

If a function is defined over half the range, say $\left(0; \frac{T}{2}\right]$, instead of the full range from $\left(-\frac{T}{2}; \frac{T}{2}\right]$, it may be expanded in a series of cosine terms only or of sine terms only. The series produced is then called a *half range Fourier series*.

The function given should be extended to the interval $\left(-\frac{T}{2}; 0\right)$ as an even or odd function. This allows the expansion of the function in a series solely of cosines (even) or sines (odd).

Suppose the function $f(x)$ is defined in the interval $\left(0; \frac{T}{2}\right]$. The even extension for this function is defined as

$$\varphi_1(x) = \begin{cases} f(x), & \text{if } x \in \left(0; \frac{T}{2}\right] \\ f(-x), & \text{if } x \in \left(-\frac{T}{2}; 0\right] \end{cases} \tag{8.44}$$

Now, if we define by $\varphi_1(x+T) = \varphi_1(x)$ the periodic extension of this even function over the whole number axis, we have the even periodic function, whose Fourier cosine series is (8.40) with coefficients computed by (8.38). In the interval $\left(0; \frac{T}{2}\right]$ (8.40) is also Fourier series for $f(x)$.

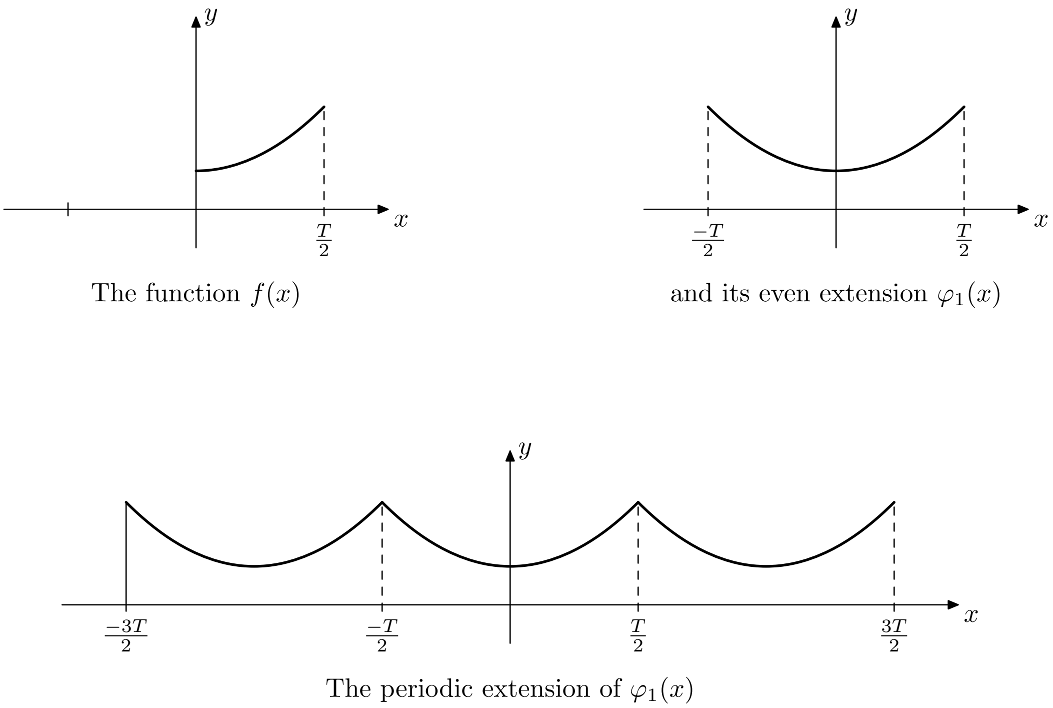
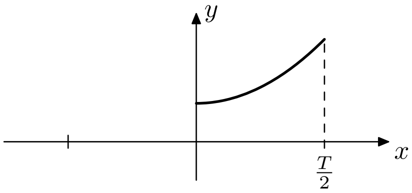


Figure 8.8.

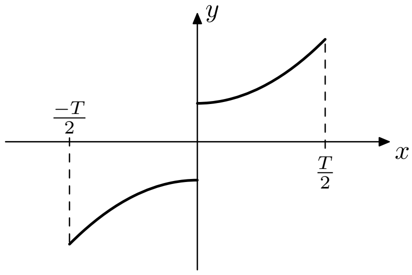
The odd extension for function $f(x)$ defined in the interval $\left(0; \frac{T}{2}\right]$ is

$$\varphi_2(x) = \begin{cases} f(x), & \text{if } x \in \left(0; \frac{T}{2}\right] \\ -f(-x), & \text{if } x \in \left(-\frac{T}{2}; 0\right] \end{cases} \tag{8.45}$$

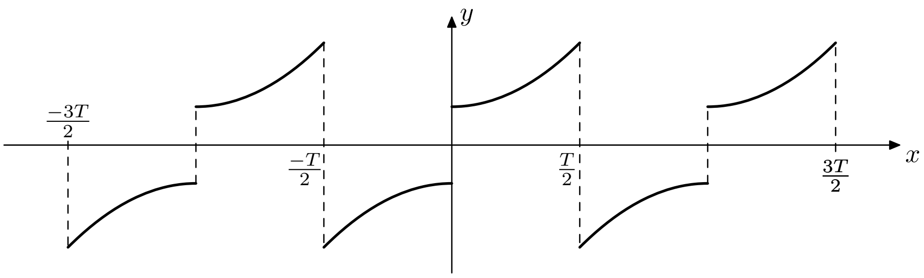
The periodic extension of $\varphi_2(x)$ defined by $\varphi_2(x+T) = \varphi_2(x)$ is an odd periodic function, whose Fourier sine series is (8.43) with coefficients computed by (8.42). In the interval $\left(0; \frac{T}{2}\right]$ (8.43) is also Fourier series for $f(x)$.



The function $f(x)$



and its odd extension $\varphi_2(x)$



The periodic extension of $\varphi_2(x)$

Figure 8.9.

Example. Find the Fourier cosine series expansion for the function defined as $f(x) = 1 - x$ in the interval $(0; 1]$.

Since $f(-x) = 1 + x$, then by (8.44) the even extension of this function in $(-1; 1]$ is

$$\varphi_1(x) = \begin{cases} 1 - x, & \text{if } \ x \in (0; 1] \\ 1 + x, & \text{if } \ x \in (-1; 0] \end{cases}$$

and the periodic extension over the whole number axis is defined by

$$\varphi_1(x + 2) = \varphi_1(x)$$

In Figure 8.10 is shown the graph of the function (red), the graph of its even extension (blue) and its periodic even extension (green). The period of the periodic extension is 2 and the frequency $\omega = \dfrac{2\pi}{2} = \pi$. By the formula

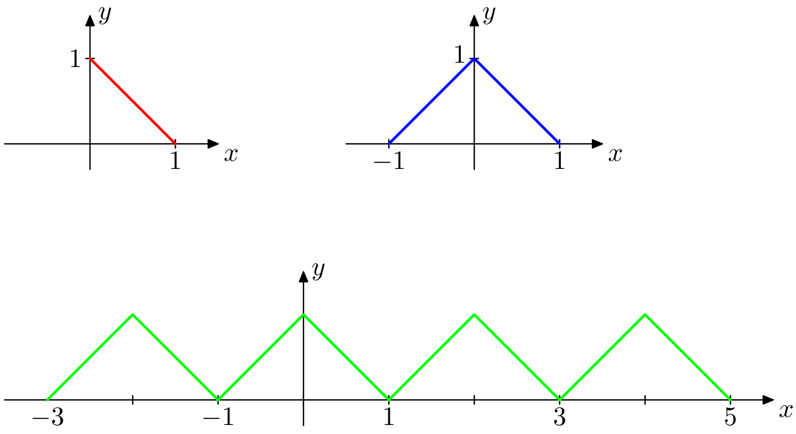


Figure 8.10.

(8.38) we compute first

$$a_0 = \frac{4}{2}\int\limits_0^1 (1-x)dx = 2\left(x - \frac{x^2}{2}\right)\Bigg|_0^1 = 1$$

and next, integrating by parts

$$\begin{aligned} a_k &= 2\int\limits_0^1 (1-x)\cos k\pi x dx = \\ &= 2\left[(1-x)\frac{1}{k\pi}\sin k\pi x\Big|_0^1 - \frac{1}{k^2\pi^2}\cos k\pi x\Big|_0^1\right] = \\ &= -\frac{2}{k^2\pi^2}((-1)^k - 1) = \begin{cases} \dfrac{4}{k^2\pi^2}, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even} \end{cases} \end{aligned}$$

Thus, the Fourier cosine series of the extension $\varphi_1(x)$ is

$$\varphi_1(x) \sim \frac{1}{2} + \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi x}{(2k+1)^2}$$

and in the interval $(0;1]$

$$1 - x \sim \frac{1}{2} + \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi x}{(2k+1)^2}$$

Since $-f(-x) = -1-x$, then by (8.45) the odd extension of this function in $(-1;1]$ is

$$\varphi_2(x) = \begin{cases} 1-x, & \text{if } x \in (0;1] \\ -1-x, & \text{if } x \in (-1;0] \end{cases}$$

and the periodic extension over the whole number axis is defined by

$$\varphi_2(x+2) = \varphi_2(x)$$

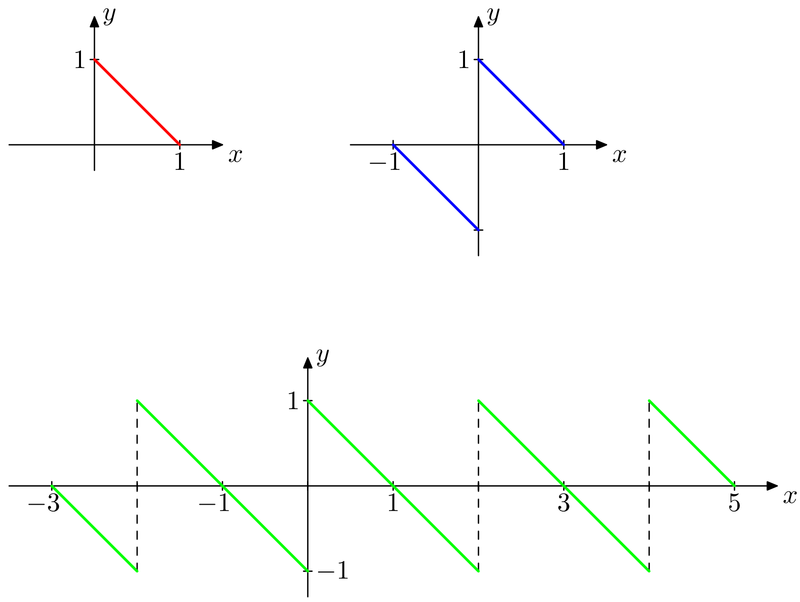


Figure 8.11.

In Figure 8.11 is shown the graph of the function (red), the graph of its odd extension (blue) and its periodic odd extension (green). The period of the periodic extension is still 2 and the frequency $\omega = \pi$.

By (8.41) the Fourier coefficients $a_k = 0$ for $k = 0,\ 1,\ 2,\ \ldots$ and by (8.42) (we integrate by parts again)

$$\begin{aligned} b_k &= 2\int_0^1 (1-x)\sin k\pi x dx = \\ &= 2\left[-(1-x)\frac{1}{k\pi}\cos k\pi x\Big|_0^1 - \frac{1}{k^2\pi^2}\sin k\pi x\Big|_0^1\right] = \frac{2}{k\pi} \end{aligned}$$

for $k = 1, 2, 3, \ldots$. Hence, the Fourier sine series of the odd periodic extension $\varphi_2(x)$ is

$$\varphi_2(x) \sim \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin k\pi x}{k}$$

and in the interval $(0; 1]$

$$1 - x \sim \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin k\pi x}{k}$$