1.2 Limit and continuity

1.2.1 Limit of sequence

There exist one-to-one correspondence between real numbers and points on numerical axis. Further we shall use in the same sense two concepts: the real number a and the point a on the numerical axis. The *distance* between two real numbers a and b (as well as the distance between two points a and b on the numerical axis) is $|b-a|$ or $|a-b|$

$$|b-a| = \begin{cases} b-a & \text{if} \quad b \geq a \\ a-b & \text{if} \quad b < a \end{cases}$$

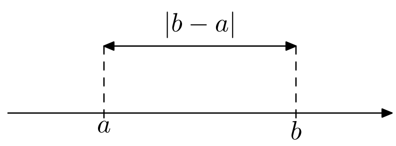

Figure 1.1: the distance between a and b

In the following definitions δ and ε are whatever positive real numbers.

Definition 1.1. A neighborhood of the point a is an arbitrary open interval $(a-\delta; a+\delta)$, which is symmetric with respect to the point a.

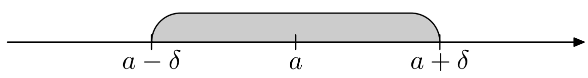

Figure 1.2: the neighborhood of a

A sequence is an infinite list of real numbers written in a specific order

$$y_1,\ y_2,\ y_3,\ \ldots,\ y_n,\ \ldots. \tag{1.1}$$

Here y_n is the nth term of the sequence and the subscript n is called the term number or index. The sequence can be treated as a function of the integer variable which associates with each integer n one and only one term of the sequence $y_n = y(n)$.

Definition 1.2. The real number b is called the *limit* of the sequence (1.1) if $\forall\ \varepsilon > 0$ there exists an integer N such that $|y_n - b| < \varepsilon$ whenever $n > N$.

Definition 1.2 tells us that in order for a limit to exist and have a finite value all the sequence (1.1) terms must be getting closer and closer to that finite value b as n increases.

In other words, the terms of the sequence get to b as close as we wish if we take n sufficiently large.

The condition in definition 1.2 $|y_n - b| < \varepsilon$ can be written $-\varepsilon < y_n - b < \varepsilon$ or

$$b - \varepsilon < y_n < b + \varepsilon$$

The last conditions are equivalent y_n belongs to the neighborhood of b i.e. $y_n \in (b - \varepsilon; b + \varepsilon)$.

Thus, we can give an equivalent definition to 1.2.

Definition 1.2'. The real number b is called the limit of the sequence (1.1) if for every neighborhood $(b - \varepsilon; b + \varepsilon)$ there exists an integer N such that $y_n \in (b - \varepsilon; b + \varepsilon)$ whenever $n > N$.

According to this definition the real number b is the limit of the sequence (1.1) if $\forall$ neighborhood $(b - \varepsilon; b + \varepsilon)$ it is possible to indicate the term of the sequence so that every following term of the sequence belongs to prescribed neighborhood of b.

Example 1.1. Considering the sequence

$$\frac{1}{2};\ \frac{2}{3};\ \frac{3}{4};\ \ldots;\ \frac{n}{n+1};\ \ldots,$$

notice, that every next term is closer to 1 than previous.

Let us determine the term of this sequence, after what all terms are closer to 1 than $\varepsilon = 0,01$, i.e. $\left|\frac{n}{n+1} - 1\right| < 0,01$.

Last condition is equivalent to $\left|\frac{n - n - 1}{n+1}\right| < 0,01$ or $\frac{1}{n+1} < 0,01$. Hence $n + 1 > 100$, i.e. $n > 99$.

Consequently, after 99th term (i.e. starting from 100th term) all the terms of this sequence are closer to 1 than 0,01.

Now determine the term, after which all the terms are closer to 1 than $\varepsilon = 0,001$, that means $\left|\frac{n}{n+1} - 1\right| < 0,001$.

As the result of similar transformations $n > 999$, i.e. after 999th term all the terms of this sequence are closer to 1 than 0,001.

Now, for an arbitrary $\varepsilon > 0$ we have the condition

$$\left|\frac{n}{n+1} - 1\right| < \varepsilon$$

or

$$\frac{1}{n+1} < \varepsilon$$

that means $n+1 > \dfrac{1}{\varepsilon}$ or $n > \dfrac{1}{\varepsilon} - 1$

In calculus we write

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

Generally, if the limit of the sequence (1.1) is b, we write

$$\lim_{n\to\infty} y_n = b$$

and we read it: the limit of the sequence (1.1) equals to b as n tends to infinity.

Example 1.2. Prove that $\lim_{n\to\infty} \dfrac{1}{n} = 0$

Let us choose an arbitrary $\varepsilon > 0$. By definition, we have the condition $\left|\dfrac{1}{n} - 0\right| < \varepsilon$, i.e. $\dfrac{1}{n} < \epsilon$ or

$$n > \frac{1}{\varepsilon}$$

We denote $[x]$ the *integer part* of the real number x and this is the greatest integer that is less than or equal to x. (For example $[4,99] = 4$, $[0,0001] = 0$, $[-2,1] = -3$ etc.)

Let $N = \left[\dfrac{1}{\varepsilon}\right]$. Then the condition $n > \dfrac{1}{\varepsilon}$ holds for every $n > N$ and $\left|\dfrac{1}{n} - 0\right| < \varepsilon$ also holds for every $n > N$, which is we wanted to prove. Notice that N depends on the choice of ε, i.e. $N = N(\varepsilon)$.

Example 1.3. A typical sequence not having a limit is

$$1;\ -1;\ 1;\ -1;\ 1;\ \dots\ ;\ (-1)^{n+1};\ \dots \tag{1.2}$$

In this sequence the terms with odd indexes equal to 1 and the terms with even indexes equal to -1.

Suppose the sequence (1.2) has the limit which is the arithmetic mean of the terms 0. Use $\varepsilon = 0,5$; let N be the corresponding integer that exists in the definition, satisfying $|(-1)^{n+1} - 0| < 0,5$ for all $n > N$. But this leads to contradiction because $|(-1)^{n+1} - 0| = 1$ for every integer value of n.

Definition 1.3. We say that a sequence of real numbers (1.1) has the limit ∞ if for every positive number M, there exists a natural number N such that if $n > N$, then $y_n > M$.

Definition 1.4. We say that a sequence of real numbers (1.1) has the limit $-\infty$ if for every negative number M, there exists a natural number N such that if $n > N$, then $y_n < M$.

1.2.2 Limit of the function

As mentioned, the sequence is a function of the integer variable and dealing with the limits of the sequence we have always the same limiting process $n \to \infty$. The limit of the function $f(x)$ can be defined for any limiting process $x \to a$, included $x \to \pm\infty$.

Definition 2.1. It is said the limit of $f(x)$ is b, as x approaches a, if $\forall$ $\varepsilon > 0$, there $\exists$ $\delta > 0$ such that for all real x, $|x-a| < \delta$ implies $|f(x)-b| < \varepsilon$.

It is written as

$$\lim_{x \to a} f(x) = b$$

and read as "the limit of f of x, as x approaches a, is b".

To say that $\lim\limits_{x \to a} f(x) = b$ means that $f(x)$ can be made as close as desired to b by making x sufficiently close, but not equal, to a.

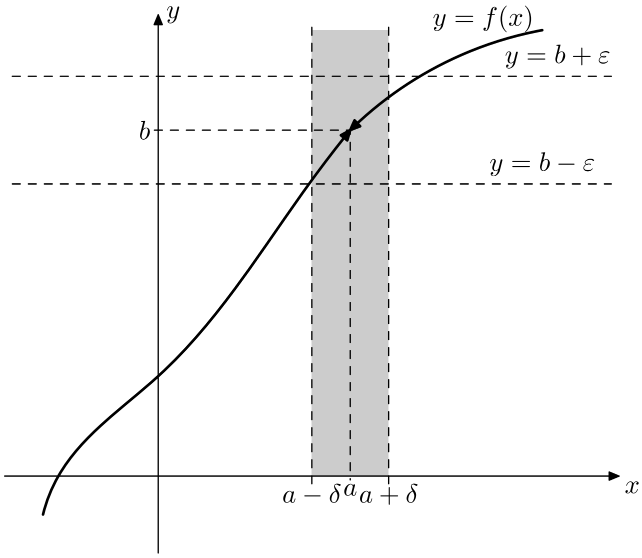

Figure 1.3: The limit of the function

It is possible to define the limit of the function in terms of neighborhoods.

Definition 2.1'. The real number b is called the limit of the function $f(x)$ as $x \to a$ if for every neighborhood $(b-\varepsilon; b+\varepsilon)$, there exists the neighborhood $(a-\delta; a+\delta)$ such that if $x \in (a-\delta; a+\delta)$ then $f(x) \in (b-\varepsilon; b+\varepsilon)$.

The definition of the limit of the function as $x \to \infty$ is actually the repetition of the limit of a sequence. Only difference is the type of the independent variable. In case of sequence n is an integer variable, in case of the function x is a real variable.

Definition 2.2. The real number b is called the limit of the function $f(x)$, as $x \to \infty$, if $\forall\ \varepsilon > 0$, there $\exists$ positive $N > 0$ such that $|f(x) - b| < \varepsilon$, whenever $x > N$.

It is written as

$$\lim_{x\to\infty} f(x) = b$$

Definition 2.3. The real number b is called the limit of the function $f(x)$, as $x \to -\infty$, if $\forall\ \varepsilon > 0$, there $\exists$ negative $N < 0$ such that $|f(x) - b| < \varepsilon$, whenever $x < N$.

It is written as

$$\lim_{x\to-\infty} f(x) = b$$

Definition 2.4. It is said that the limit of the function $f(x)$ is ∞, as $x \to a$, if $\forall\ N > 0$ there $\exists\ \delta > 0$ such that $f(x) > N$, whenever $|x - a| < \delta$.

It is written as

$$\lim_{x\to a} f(x) = \infty$$

Definition 2.5. It is said that the limit of the function $f(x)$ is $-\infty$, as $x \to a$, if $\forall\ N > 0$ there $\exists\ \delta > 0$ such that $f(x) < -N$, whenever $|x - a| < \delta$.

It is written as

$$\lim_{x\to a} f(x) = -\infty$$

Let $N > 0$. The neighborhood of ∞ is arbitrary open interval $(N; \infty)$ and the neighborhood of $-\infty$ is arbitrary open interval $(-\infty; -N)$.

Using the neighborhoods we can give alternative definitions to the definitions 2.2, 2.3, 2.4 and 2.5. For example, the alternative definition to the definition 2.4.

Definition 2.4'. It is said that the limit of the function $f(x)$ is ∞, as $x \to a$, if for each neighborhood $(N; \infty)$ there exists a neighborhood $(a - \delta; a + \delta)$ such that $f(x) \in (N; \infty)$, whenever $x \in (a - \delta; a + \delta)$.

1.2.3 One-sided limits

Often it is not possible to describe the function's behavior with a single limit. We can describe the function's behavior from the right and from the left using two limits.

The left-hand neighborhood of the point a is an arbitrary open interval $(a-\varepsilon; a)$ and the right-hand neighborhood of the point a is an arbitrary open interval $(a; a + \varepsilon)$.

Definition 3.1. The real number b_1 is called the left-hand limit of the function $f(x)$, as x approaches a from the left, if $\forall$ neighborhood b_1 $(b_1 - \varepsilon; b_1 + \varepsilon)$ there $\exists$ a left-hand neighborhood of a $(a - \delta; a)$ such that $f(x) \in (b_1 - \varepsilon; b_1 + \varepsilon)$, whenever $x \in (a - \delta; a)$.

It is written

$$\lim_{x \to a-} f(x) = b_1$$

and it is read: the limit of the function $f(x)$, as x approaches a from the left, equals to b_1.

Definition 3.2. The real number b_2 is called the right-hand limit of the function $f(x)$, as x approaches a from the right, if $\forall$ neighborhood b_2 $(b_2 - \varepsilon; b_2 + \varepsilon)$ there $\exists$ a right-hand neighborhood of a $(a; a + \delta)$ such that $f(x) \in (b_2 - \varepsilon; b_2 + \varepsilon)$, whenever$x \in (a; a + \delta)$.

It is written

$$\lim_{x \to a+} f(x) = b_2$$

and it is read: the limit of the function $f(x)$, as x approaches a from the right, equals to b_2.

Theorem 3.1. If the function has the limit, as $x \to a$, then it has both one-sided limits and these one-sided limits are equal.

Proof. Assume $\lim\limits_{x \to a} f(x) = b$. According to the definition of the limit of the function $\forall$ neighborhood of b $(b - \varepsilon; b + \varepsilon)$ there exists a neighborhood of a $(a - \delta; a + \delta)$ such that if $x \in (a - \delta; a + \delta)$, then $f(x) \in (b - \varepsilon; b + \varepsilon)$. But in both cases if $x \in (a - \delta; a)$ and if $x \in (a; a + \delta)$ we have $x \in (a - \delta; a + \delta)$. Thus, for every neighborhood $(b - \varepsilon; b + \varepsilon)$ there exists a left-hand neighborhood of a $(a - \delta; a)$ such that if $x \in (a - \delta; a)$, then $f(x) \in (b - \varepsilon; b + \varepsilon)$. As well there exists a right-hand neighborhood of a $(a; a + \delta)$ such that if $x \in (a; a + \delta)$, then $f(x) \in (b - \varepsilon; b + \varepsilon)$. According to the definitions of the one-sided limits

$$\lim_{x \to a-} f(x) = b$$

and

$$\lim_{x \to a+} f(x) = b$$

Theorem 3.2. If the function $f(x)$ has one-sided limits and these one-sided limits are equal, then this function has the limit (and this equals to the one-sided limits).

If one-sided limits of the function $f(x)$ exist, but are not equal, i.e.

$$\lim_{x \to a-} f(x) \neq \lim_{x \to a+} f(x)$$

then this function has no limit, as $x \to a$.

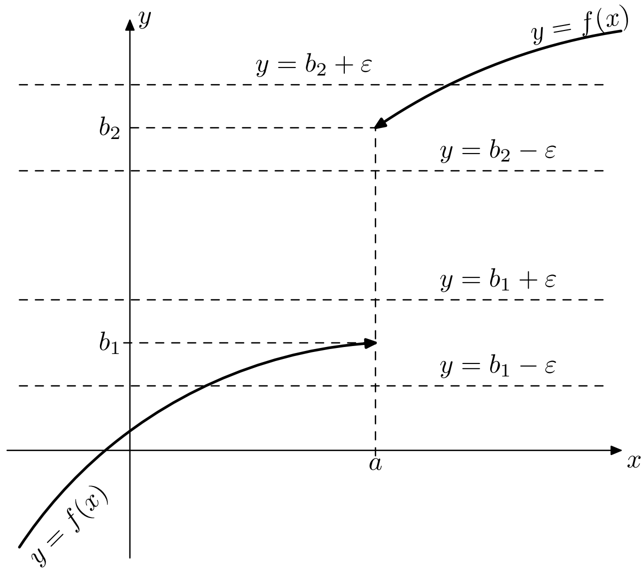

Figure 1.4: One-sided limits

Example 3.1. Let us find one-sided limits of the function $f(x) = \dfrac{|x|}{x}$, as $x \to 0$: $\lim\limits_{x\to 0-} \dfrac{|x|}{x}$ and $\lim\limits_{x\to 0+} \dfrac{|x|}{x}$.

If $x \to 0-$, then $x < 0$ and $|x| = -x$, i.e. $\dfrac{|x|}{x} = -1$. The limit of the constant function equals to that constant, thus,

$$\lim_{x\to 0-} \frac{|x|}{x} = -1$$

If $x \to 0+$, then $x > 0$ and $|x| = x$, i.e. $\dfrac{|x|}{x} = 1$. Thus,

$$\lim_{x\to 0+} \frac{|x|}{x} = 1$$

One-sided limits are different, it follows that there does not exist a limit

$$\lim_{x\to 0} \frac{|x|}{x}$$

Example 3.2. Let us find $\lim\limits_{x\to 0-} \arctan \dfrac{1}{x}$ and $\lim\limits_{x\to 0+} \arctan \dfrac{1}{x}$.

If $x \to 0-$, then $\dfrac{1}{x} \to -\infty$, hence

$$\lim_{x\to 0-} \arctan \frac{1}{x} = -\frac{\pi}{2}$$

If $x \to 0+$, then $\dfrac{1}{x} \to \infty$ and

$$\lim_{x\to 0-} \arctan \frac{1}{x} = \frac{\pi}{2}$$

We conclude that there exists no limit $\lim\limits_{x\to 0} \arctan \dfrac{1}{x}$.

Example 3.3. The function $\dfrac{\sin x}{x}$ has no value at the point $x \neq 0$.

It is possible to prove that $\lim\limits_{x\to 0-} \dfrac{\sin x}{x} = 1$ and $\lim\limits_{x\to 0+} \dfrac{\sin x}{x} = 1$. Consequently there $\exists$ the limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

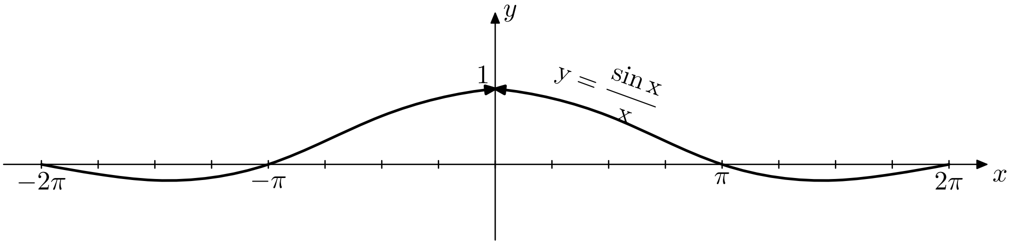

Figure 1.5: The function $y = \dfrac{\sin x}{x}$

1.2.4 Infinite quantities and infinitesimals

Consider the dependent variable $y = y(x)$ and the limiting process $x \to a$ (included $x \to \pm\infty$).

Definition 4.1. The variable y is called *infinitely large*, as $x \to a$, if

$$\lim_{x\to a} |y| = \infty$$

that means $\lim\limits_{x\to a} y = \infty$ or $\lim\limits_{x\to a} y = -\infty$

Example 4.1. The function $y = \dfrac{1}{x}$ is infinitely large, as $x \to 0$, because

$$\lim_{x\to 0-} \frac{1}{x} = -\infty$$

and

$$\lim_{x\to 0+} \frac{1}{x} = \infty$$

Example 4.2. The function $y = \ln x$ is infinitely large, as $x \to 0+$, because

$$\lim_{x\to 0+} \ln x = -\infty$$

and as $x \to \infty$, because

$$\lim_{x\to \infty} \ln x = \infty$$

Definition 4.2. The variable $\alpha = \alpha(x)$ is called *infinitesimal*, as $x \to a$, if $\lim\limits_{x\to a} \alpha = 0$.

The next theorem is useful in many following proofs.

Theorem 4.1. The limit of the variable y equals to b if and only if it is the sum of b and an infinitesimal α, i.e.

$$\lim_{x\to a} y = b \iff y = b + \alpha$$

Proof. Necessity. Suppose that $\lim\limits_{x\to a} y = b$, i.e. $\forall\ \varepsilon > 0$ there $\exists\ \delta > 0$ such that if $|x - a| < \delta$, then $|y - b| < \varepsilon$.

Denoting $\alpha = y - b$ we have $y = b + \alpha$ and $\forall\ \varepsilon > 0$ there $\exists\ \delta > 0$ such that $|x - a| < \delta$, then $|\alpha| < \varepsilon$. Due to the definition of the limit we have $\lim\limits_{x\to a} \alpha = 0$, i.e. α is an infinitesimal.

Sufficiency Suppose that $y = b + \alpha$, where α is an infinitesimal. Then $\alpha = y - b$ and, as α is an infinitesimal, then $\forall\ \ \varepsilon > 0$ there $\exists\ \ \delta > 0$ such that if $|x - a| < \delta$, then $|y - b| < \varepsilon$, which means that the limit of y, as x approaches a, equals to b.

Theorem 4.2. The sum of two infinitesimals is an infinitesimal, i.e. if α and β are two infinitesimals, then $\alpha + \beta$ is an infinitesimal.

Proof. The variable α is an infinitesimal, as $x \to a$, i.e. $\forall\ \varepsilon > 0$ there $\exists$ $\delta_1 > 0$ such that

$$|\alpha| < \frac{\varepsilon}{2}$$

whenever $|x - a| < \delta_1$.

The variable β is an infinitesimal too, as $x \to a$, i.e. $\forall\ \varepsilon > 0$ there $\exists$ $\delta_2 > 0$ such that

$$|\beta| < \frac{\varepsilon}{2}$$

whenever $|x - a| < \delta_2$.

If we choose δ to be the least of two positive real numbers δ_1 and δ_2, i.e. $\delta = \min\{\delta_1, \delta_2\}$, we have that if $|x - a| < \delta$, then

$$|\alpha + \beta| \le |\alpha| + |\beta| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

That means, taking the value of x sufficiently close to a, $\alpha+\beta$ is as close to zero as we wish, i.e $\alpha+\beta$ is an infinitesimal.

Remark. The theorem 4.2 holds also if we add three, four, ... infinitesimals.

For example, if we have three infinitesimals α, β and γ, then we can write

$$\alpha+\beta+\gamma=(\alpha+\beta)+\gamma$$

By the theorem 4.2 $\alpha+\beta$ is an infinitesimal and by this theorem, again, $(\alpha+\beta)+\gamma$ is an infinitesimal as the sum of two infinitesimals.

Definition 4.3. The variable y is called *bounded* in the neighborhood of a $(a-\delta; a+\delta)$ if there $\exists$ a constant $N>0$ such that

$$|y|<N$$

whenever $x \in (a-\delta; a+\delta)$.

Theorem 4.3. The product αy of a bounded variable y and an infinitesimal α is an infinitesimal.

Proof. As y is bounded, then for some neighborhood of a, $(a-\delta_1; a+\delta_1)$ there $\exists$ a constant $N>0$ such that $|y|<N$. The variable α is an infinitesimal, i.e. $\forall\ \varepsilon>0$ there $\exists\ \delta_2>0$ such that $|\alpha|<\dfrac{\varepsilon}{N}$, whenever $|x-a|<\delta_2$. If we choose $\delta=\min\{\delta_1,\delta_2\}$ and assume that $|x-a|<\delta$, then

$$|\alpha y|=|\alpha||y|<\frac{\varepsilon}{N}\cdot N=\varepsilon$$

i.e. αy is an infinitesimal.

Conclusion 4.4. The product of a constant and an infinitesimal is an infinitesimal.

This is the direct conclusion of the theorem 4.3 because any constant is bounded.

Conclusion 4.5. The product of two infinitesimals is an infinitesimal, i.e. if α and β are two infinitesimals, then $\alpha\beta$ is an infinitesimal.

Proof. An infinitesimal, as $x \to a$, is bounded in the neighborhood of a (and bounded with very small real number ε).

Theorem 4.6. The quotient of an infinitesimal and a variable with non-zero limit is an infinitesimal, i.e. if α is an infinitesimal and $\lim\limits_{x\to a} y=b$ with $b\neq 0$, then $\dfrac{\alpha}{y}$ is an infinitesimal.

Proof. In this proof we use the property of absolute value of the real numbers $||a|-|b|| \leq |a-b|$.

If $\lim\limits_{x\to a} y=b$, then $\forall\ \varepsilon>0$ there $\exists\ \delta>0$ such that $|y-b|<\varepsilon$, whenever $|x-a|<\delta$.

By the property of the absolute value mentioned above $||y| - |b|| < \varepsilon$ or $-\varepsilon < |y| - |b| < \varepsilon$, hence

$$|b| - \varepsilon < |y| < |b| + \varepsilon$$

Thus, for reciprocals

$$\frac{1}{|b| + \varepsilon} < \frac{1}{|y|} < \frac{1}{|b| - \varepsilon}$$

As ε is whatever positive number, we can choose it $\varepsilon = 0,1|b|$, for which

$$\frac{1}{1,1|b|} < \frac{1}{|y|} < \frac{1}{0,9|b|}$$

that means $\frac{1}{y}$ is a bounded variable the product

$$\frac{\alpha}{y} = \alpha \cdot \frac{1}{y}$$

as the product of an infinitesimal and a bounded variable is due to theorem 4.3 an infinitesimal.

Remark. The quotient of two infinitesimals will be in detail discussed in subsection 1.2.7.

1.2.5 Limit theorems

There exist two types of limit theorems: the theorems connected with arithmetical operations and the theorems connected with ordering.

We start from limit theorems connected with arithmetical operations. Suppose two functions $y = y(x)$ and $z = z(x)$ are defined in some neighborhood of the point a and there exist the limits $\lim\limits_{x \to a} y$ and $\lim\limits_{x \to a} z$.

Theorem 5.1. The limit of the sum of two variables equals to the sum of limits of these variables:

$$\lim_{x \to a}(y + z) = \lim_{x \to a} y + \lim_{x \to a} z$$

Proof. We shall denote

$$\lim_{x \to a} y = b_1$$

and

$$\lim_{x \to a} z = b_2$$

By the theorem 4.1 there exist two infinitesimals α and β such that $y = b_1 + \alpha$ and $z = b_2 + \beta$. Therefore $y + z = b_1 + b_2 + \alpha + \beta$.

Due to the theorem 4.2 $\alpha + \beta$ is an infinitesimal, thus using once more the theorem 4.1, we have

$$\lim_{x \to a}(y + z) = b_1 + b_2$$

which is we wanted to prove.

Theorem 5.2. The limit of the product of two variables equals to the product of the limits of these variables:

$$\lim_{x \to a} yz = \lim_{x \to a} y \cdot \lim_{x \to a} z$$

Proof. Let us denote again

$$\lim_{x \to a} y = b_1$$

and

$$\lim_{x \to a} z = b_2$$

Due to the theorem 4.1, there exist the infinitesimals α and β such that $y = b_1 + \alpha$ and $z = b_2 + \beta$. Therefore $yz = (b_1 + \alpha)(b_2 + \beta)$, i.e. $yz = b_1b_2 + b_1\beta + \alpha b_2 + \alpha\beta$.

According to the conclusion 4.4 $b_1\beta$ and αb_2 infinitesimals (as the products of the constant and an infinitesimal). Due to conclusion 4.5 $\alpha\beta$ is an infinitesimal. By the theorem 4.2 $\gamma = b_1\beta + \alpha b_2 + \alpha\beta$ is an infinitesimal. Thus,

$$yz = b_1b_2 + \gamma$$

where γ is an infinitesimal and using the theorem 4.1 we conclude that

$$\lim_{x \to a} yz = b_1b_2$$

what is we wanted to prove.

Conclusion 5.3. The constant coefficient can be carried outside the limit, i.e. if c is a constant, then

$$\lim_{x \to a} cy = c \lim_{x \to a} y$$

Proof. We use the previous theorem

$$\lim_{x \to a} cy = \lim_{x \to a} c \cdot \lim_{x \to a} y$$

and that the limit of a constant equals to this constant $\lim\limits_{x \to a} c = c$.

Conclusion 5.4. The limit of the difference of two variables equals to the difference of the limits of these variables:

$$\lim_{x\to a}(y-z)=\lim_{x\to a}y-\lim_{x\to a}z$$

To prove we write

$$\lim_{x\to a}(y-z)=\lim_{x\to a}(y+(-1)z)$$

By theorem 5.1

$$\lim_{x\to a}(y+(-1)z)=\lim_{x\to a}y+\lim_{x\to a}(-1)z$$

and by conclusion 5.3

$$\lim_{x\to a}y+\lim_{x\to a}(-1)z=\lim_{x\to a}y-\lim_{x\to a}z$$

Theorem 5.5. The limit of the quotient of two variables equals to the quotient of the limits of these variables if the limit of the divisor does not equal to zero:

$$\lim_{x\to a}\frac{y}{z}=\frac{\lim\limits_{x\to a}y}{\lim\limits_{x\to a}z}$$

if $\lim\limits_{x\to a}z\neq 0$.

Proof. Let us denote again

$$\lim_{x\to a}y=b_1$$

and

$$\lim_{x\to a}z=b_2\neq 0$$

By theorem 4.1 there exist two infinitesimals α and β such that $y=b_1+\alpha$ and $z=b_2+\beta$.

Then

$$\frac{y}{z}=\frac{b_1+\alpha}{b_2+\beta}=\frac{b_1}{b_2}+\frac{b_1+\alpha}{b_2+\beta}-\frac{b_1}{b_2}$$

Taking two last fractions to common denominator

$$\frac{y}{z}=\frac{b_1}{b_2}+\frac{b_1b_2+\alpha b_2-b_1b_2-\beta b_1}{b_2(b_2+\beta)}=\frac{b_1}{b_2}+\frac{\alpha b_2-\beta b_1}{b_2(b_2+\beta)} \tag{1.3}$$

The numerator of the last fraction $\alpha b_2+(-b_1)\beta$ is an infinitesimal by conclusion 4.4 and theorem 4.2. The denominator of this fraction $b_2^2+b_2\beta$ is the sum of the constant b_2^2 and an infinitesimal $b_2\beta$. By theorem 4.1 the limit of the denominator $b_2^2 \neq 0$. The fraction

$$\frac{\alpha b_2 - \beta b_1}{b_2(b_2+\beta)}$$

is a quotient of an infinitesimal and a variable which has non-zero limit. By theorem 4.6 this is an infinitesimal. Hence in (1.3) the quotient $\frac{y}{z}$ can be expressed as the sum of the constant $\frac{b_1}{b_2}$ and an infinitesimal $\frac{\alpha b_2 - \beta b_1}{b_2(b_2+\beta)}$. Using theorem 4.1 we conclude that

$$\lim_{x\to a} \frac{y}{z} = \frac{b_1}{b_2},$$

which is we wanted to prove.

Now we proceed with limit theorems connected with ordering.

Theorem 5.6. The limit of the non-negative variable is non-negative, i.e. if $y \geq 0$ in some neighborhood of a and there $\exists \lim_{x\to a} y = b$ then $b \geq 0$.

Proof. Let us assume that the opposite assertion $\lim_{x\to a} y = b < 0$ holds. If $y \geq 0$ and $b < 0$ then $|y-b| > |b|$. If we choose positive ε such that $\varepsilon < |b|$ then the condition $|y-b| < \varepsilon$ cannot be satisfied in any neighborhood of a which leads to the contradiction with respect to assumption $\lim_{x\to a} y = b$. We got this contradiction because of antithesis $b < 0$, consequently $\lim_{x\to a} y = b \geq 0$

Theorem 5.7. If in some neighborhood of a holds $y \geq z$ and there exist the limits $\lim_{x\to a} y$ and $\lim_{x\to a} z$ then $\lim_{x\to a} y \geq \lim_{x\to a} z$.

Proof. If $y \geq z$ then $y - z \geq 0$. By theorem 5.6 $\lim_{x\to a}(y-z) \geq 0$ and by conclusion 5.4 $\lim_{x\to a} y - \lim_{x\to a} z \geq 0$ which proves the assertion.

In the next theorem we use three variables $u = u(x)$, $v = v(x)$ and $w = w(x)$.

Theorem 5.8. If in some neighborhood of a $u \leq w \leq v$, there $\exists$ equal limits $\lim_{x\to a} u = b$ and $\lim_{x\to a} v = b$ then $\lim_{x\to a} w = b$.

Proof. If $u \leq w$ and $w \leq v$ then by theorem 5.7 $\lim_{x\to a} u \leq \lim_{x\to a} w$ and $\lim_{x\to a} w \leq \lim_{x\to a} v$. As we have assumed $\lim_{x\to a} u = b$ and $\lim_{x\to a} v = b$. Thus,

$$b \leq \lim_{x\to a} w \leq b$$

that means $\lim_{x\to a} w = b$, quod erat demonstrandum.

Theorem 5.9. Monotonically increasing (decreasing) bounded variable has a finite limit if $x \to \pm\infty$.

The proof of this theorem we skip because the to prove it we need some untreated facts of the theory of real numbers. But here is an example.

Example 5.1. The function $y = \arctan x$ is increasing as $x \to \infty$ and bounded because $\forall x \in \mathbb{R}$ there holds $|\arctan x| < \frac{\pi}{2}$. The limit

$$\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$$

One important conclusion from Theorem 5.9 is about the limit of the sequence.

Conclusion Monotonically increasing (decreasing) bounded sequence has a finite limit.

1.2.6 The limit $\lim_{x\to 0} \frac{\sin x}{x}$

We use theorem 5.8 to prove the formula

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{1.4}$$

First assume $x > 0$. Because of $x \to 0$ we can impose the restriction $0 < x < \frac{\pi}{2}$, i.e. x is an acute angle. Let us draw the unit circle, triangle OPQ with acute angle x, sector OPQ with central angle x and right triangle OPR with acute angle x (see Figure 1.6). It is obvious that the triangle OPQ is

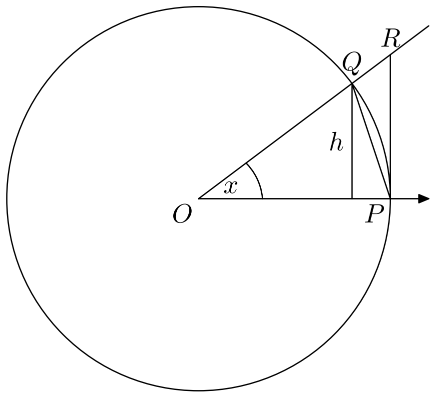

Figure 1.6:

included in sector OPQ and this sector is included in right triangle OPR. Therefore the areas satisfy inequalities

$$S_{\triangle OPQ} < S_{sector OPQ} < S_{\triangle OPR}$$

Denoting the height of the triangle OPQ by h and using the fact we have unit circle, i.e. $OP = OQ = 1$ we obtain

$$\frac{1 \cdot h}{2} < \frac{x \cdot 1^2}{2} < \frac{1 \cdot PR}{2}$$

But $\sin x = \frac{h}{1}$ and $\tan x = \frac{PR}{1}$, hence $h = \sin x$ and $PR = \tan x = \frac{\sin x}{\cos x}$. Replacing these into inequalities, we have

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\sin x}{2 \cos x}$$

Now we multiply these inequalities by 2 and divide by $\sin x$ (which is positive because x is an acute angle). As the result we have

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

The reciprocals satisfy opposite inequalities

$$\cos x < \frac{\sin x}{x} < 1$$

As $\lim_{x \to 0} \cos x = 1$ then by theorem 5.8

$$\lim_{x \to 0+} \frac{\sin x}{x} = 1$$

If $x < 0$ then $-x > 0$ and, using what we have just proved,

$$\lim_{-x \to 0+} \frac{\sin(-x)}{-x} = 1$$

But $\sin(-x) = -\sin x$ and if $-x \to 0+$ then $x \to 0-$, thus,

$$\lim_{x \to 0-} \frac{\sin x}{x} = 1$$

One-sided limits are equal, consequently there holds (1.4).

Example 6.1. Find $\lim_{x \to 0} \frac{x}{\sin x}$

Writing this limit $\lim_{x \to 0} \frac{1}{\frac{\sin x}{x}}$, we obtain by theorem 5.5 that $\frac{1}{\lim_{x \to 0} \frac{\sin x}{x}} = \frac{1}{1} = 1$.

Example 6.2. Find $\lim\limits_{x\to 0}\dfrac{\arcsin x}{x}$.

Let $t=\arcsin x$. From $x\to 0$ it follows that $t\to 0$ and $x=\sin t$. Using example 6.1

$$\lim_{x\to 0}\frac{\arcsin x}{x}=\lim_{t\to 0}\frac{t}{\sin t}=1$$

Example 6.3 Find $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 4x}$.

If we divide numerator and denominator by x (we can do it because $x\to 0$ that means $x\neq 0$), we obtain

$$\lim_{x\to 0}\frac{\dfrac{\sin 3x}{x}}{\dfrac{\sin 4x}{x}}$$

Multiplying the numerator and denominator of the fraction $\dfrac{\sin 3x}{x}$ by 3 and the numerator and denominator of the fraction $\dfrac{\sin 4x}{x}$ by 4 we have

$$\lim_{x\to 0}\frac{\dfrac{3\sin 3x}{3x}}{\dfrac{4\sin 4x}{4x}}=\frac{3}{4}\lim_{x\to 0}\frac{\dfrac{\sin 3x}{3x}}{\dfrac{\sin 4x}{4x}}$$

By theorem 5.5

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 4x}=\frac{3}{4}\frac{\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}}{\lim\limits_{x\to 0}\dfrac{\sin 4x}{4x}}=\frac{3}{4}$$

because if $x\to 0$ then $3x\to 0$ and $4x\to 0$.

Example 6.4 Find $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$.

Here we multiply the numerator and denominator by $1+\cos x$. We obtain

$$\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}=\lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}=\lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}$$

Writing the last fraction as the product of three factors and applying theorem 5.2

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{1+\cos x}=1\cdot 1\cdot\frac{1}{2}=\frac{1}{2}$$

1.2.7 Real number e

Let us consider the sequence with general term $y_n = \left(1+\frac{1}{n}\right)^n$, i.e. the sequence

$$2,\ \frac{9}{4},\ \frac{64}{27},\ \frac{625}{256},\ \ldots,\ \left(1+\frac{1}{n}\right)^n,\ldots \tag{1.5}$$

We shall prove that this sequence is bounded and increasing. By Newton's binomial formula

$$y_n = \left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \ldots + \frac{n(n-1)\ldots(n-k+1)}{k!}\frac{1}{n^k} + \ldots + \frac{1}{n^n} =$$
$$1+1+\frac{1}{2!}(1-\frac{1}{n}) + \ldots + \frac{1}{k!}(1-\frac{1}{n})\ldots(1-\frac{k-1}{n}) + \ldots + \frac{1}{n^n} <$$
$$2+\frac{1}{2!}+\ldots\frac{1}{k!}+\ldots\frac{1}{n!}$$

If $k \geq 2$ then $\frac{1}{k!} \leq \frac{1}{2^{k-1}}$, Therefore

$$y_n \leq 2 + \frac{1}{2} + \ldots + \frac{1}{2^{k-1}} + \ldots \frac{1}{2^{n-1}} = 2 + \frac{\frac{1}{2}(1-(\frac{1}{2})^{n-1})}{\frac{1}{2}} < 3$$

that is, the sequence is bounded. Using transformations for y_n, we obtain

$$y_{n+1} = 2+\frac{1}{2!}(1-\frac{1}{n+1})+\ldots+\frac{1}{k!}(1-\frac{1}{n+1})\ldots(1-\frac{k-1}{n+1})+\ldots+\frac{1}{(n+1)^{n+1}}$$

The inequality $\frac{i}{n} > \frac{i}{n+1}$ $i = 1,\ldots,k-1$ yields $1-\frac{i}{n} < 1-\frac{i}{n+1}$. Then in the expansion by binomial formula

$$\frac{1}{k!}(1-\frac{1}{n})\ldots(1-\frac{k-1}{n}) < \frac{1}{k!}(1-\frac{1}{n+1})\ldots(1-\frac{k-1}{n+1})$$

that is, in the expansion of y_n each corresponding term is less than in the expansion of y_{n+1}. In addition, in the expansion of y_{n+1} there is one extra positive term $\frac{1}{(n+1)^{n+1}}$, that means $y_n < y_{n+1}$ or the sequence (1.5) is increasing.

According to theorem 5.9 the sequence (1.5) has a limit. This limit is called *Euler's number* and denoted

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

If $x > 0$ is a arbitrary real number, then there exists an integer n such that $n \leq x < n+1$. It follows

$$\frac{1}{n+1} < \frac{1}{x} \leq \frac{1}{n}$$

or

$$1 + \frac{1}{n+1} < 1 + \frac{1}{x} \leq 1 + \frac{1}{n}$$

consequently

$$\left(1 + \frac{1}{n+1}\right)^n < \left(1 + \frac{1}{x}\right)^x < \left(1 + \frac{1}{n}\right)^{n+1} \tag{1.6}$$

When $n \to \infty$ then, as $n \leq x \leq n+1$, it follows that $x \to \infty$ and $n+1 \to \infty$.

Now we find the limits

$$\lim_{n\to\infty} \left(1 + \frac{1}{n+1}\right)^n = \lim_{n\to\infty} \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{1 + \frac{1}{n+1}} = \frac{e}{1} = e$$

and

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n+1} = \lim_{n\to\infty} \left[\left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)\right] = e \cdot 1 = e$$

Due to the conditions (1.6) we conclude by theorem 5.8 that

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$$

We find also the limit $\lim\limits_{x\to-\infty} \left(1 + \frac{1}{x}\right)^x$. For this we use the change of variable $x = -1 - t$. When $x \to -\infty$ then $t \to \infty$. Now we evaluate

$$\lim_{x\to-\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{t\to\infty} \left(1 + \frac{1}{-1-t}\right)^{-1-t} =$$

$$\lim_{t\to\infty} \left(1 - \frac{1}{1+t}\right)^{-(1+t)} = \lim_{t\to\infty} \left(\frac{1+t-1}{1+t}\right)^{-(1+t)} =$$

$$\lim_{t\to\infty} \left(\frac{1+t}{t}\right)^{1+t} = \lim_{t\to\infty} \left[\left(1 + \frac{1}{t}\right)^t \left(1 + \frac{1}{t}\right)\right] = e$$

Hence in both limiting processes $x \to \infty$ and $x \to -\infty$ the limit of the function $y = \left(1 + \frac{1}{x}\right)^x$ equals to e, i.e.

$$\lim_{x\to\pm\infty} \left(1 + \frac{1}{x}\right)^x = e \tag{1.7}$$

One uses the last formula if there is indeterminacy 1^∞. The constant e is an irrational number and its numerical value truncated to 12 decimal places is $2,718281828459....$

Example 6.1. Find $\lim\limits_{x\to\infty}\left(\dfrac{2x+3}{2x-1}\right)^x$.

We use the transformations $\dfrac{2x+3}{2x-1}=\dfrac{2x-1+4}{2x-1}=1+\dfrac{4}{2x-1}=1+\dfrac{1}{\frac{2x-1}{4}}$ and the change of variable $t=\dfrac{2x-1}{4}$. When $x\to\infty$, then $t\to\infty$ and if we express x in terms of new variable t, we have $x=2t+\dfrac{1}{2}$. Thus

$$\lim_{x\to\infty}\left(\frac{2x+3}{2x-1}\right)^x=\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{2t+\frac{1}{2}}=\lim_{t\to\infty}\left[\left(1+\frac{1}{t}\right)^t\right]^2\left(1+\frac{1}{t}\right)^{\frac{1}{2}}=e^2$$

Example 6.2. Find $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}$.

If $x\to 0+$ then $\dfrac{1}{x}\to\infty$ and if $x\to 0-$ then $\dfrac{1}{x}\to-\infty$. Hence, by (1.7)

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=\lim_{\frac{1}{x}\to\pm\infty}\ln\left(1+\frac{1}{\frac{1}{x}}\right)^{\frac{1}{x}}=\ln e=1$$

1.2.8 Comparison of infinitesimals

Let $=\alpha(x)$ and $\beta=\beta(x)$ be two infinitesimals as $x\to a$. Although the limit of both variables equal to 0, the speed by which these variables approach to 0 can be very different. This is the reason we have to compare infinitesimals. We compare two infinitesimals by evaluating of the limit of the ratio of these infinitesimals

$$\lim_{x\to a}\frac{\alpha(x)}{\beta(x)}$$

Definition 7.1. If

$$\lim_{x\to a}\frac{\alpha(x)}{\beta(x)}=0$$

then α is said to be an infinitesimal of a higher order with respect to β as $x\to a$.

Also it is said that β is an infinitesimal of a lower order with respect to α.

Example 7.1. As $x \to 0$ then $\alpha = x^3$ is an infinitesimal of a higher order with respect to $\beta = x^2$ because

$$\lim_{x\to 0} \frac{x^3}{x^2} = \lim_{x\to 0} x = 0$$

Definition 7.2. If

$$\lim_{x\to a} \frac{\alpha(x)}{\beta(x)} = \infty$$

then α is said to be an infinitesimal of a lower order with respect to β as $x \to a$.

Definition 7.3. If

$$\lim_{x\to a} \frac{\alpha(x)}{\beta(x)} = b \neq 0$$

then it is said that α and β are infinitesimals of the same order as $x \to a$.

Example 7.2. Two variables $\alpha = 1 - \cos x$ and $\beta = x^2$ are infinitesimals of the same order as $x \to 0$ because according to example 6.4

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

Definition 7.4. If

$$\lim_{x\to a} \frac{\alpha(x)}{\beta(x)} = 1$$

then it is said that α and β are equivalent infinitesimals as $x \to a$ and it is written

$$\alpha \sim \beta$$

Example 7.3. As $x \to 0$ the variables $\alpha = \sin x$ and $\beta = x$ are equivalent infinitesimals because

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Example 7.4. As $x \to 0$ the variables $\alpha = \ln(1 + x)$ and $\beta = x$ are equivalent infinitesimals because (Example 6.2)

$$\lim_{x\to 0} \frac{\ln(1 + x)}{x} = 1$$

1.2.9 Continuity of function

Definition 8.1. The function $y = f(x)$ is said to be continuous at a point a if the following conditions are satisfied

1. $\exists\ f(a)$

2. $\exists \lim_{x \to a} f(x)$

3. $\lim_{x \to a} f(x) = f(a)$

Definition 8.2. The function $y = f(x)$ is said to be continuous on an interval $(a; b)$ if it is continuous at each point in this interval. The function $y = f(x)$ is said to be continuous if it is continuous at every point in its domain.

Example 8.1. The function $f(x) = \dfrac{\sin x}{x}$ **is not continuous** at the point $x = 0$ because $f(0)$ is undefined.

The function

$$g(x) = \begin{cases} \frac{\sin x}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

is continuous at the point $x = 0$ because $g(0) = 1$ and

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$$

and all three conditions at the point $x = 0$ are satisfied.

The function

$$h(x) = \begin{cases} \frac{\sin x}{x} & \text{if} \quad x \neq 0 \\ 2 & \text{if} \quad x = 0 \end{cases}$$

is not continuous at the point $x = 0$ because $h(0) = 2$ and

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$$

and third condition of the continuity at the point $x = 0$ is not satisfied.

In further consideration we denote the fixed point by x and the variable by $x + \Delta x$. We say that Δx is an increment of the independent variable x. Then the limiting process $x + \Delta x \to x$ is equivalent to the process $\Delta x \to 0$. Provided two first conditions of continuity are satisfied the third condition of continuity at the point x is written $\lim_{\Delta x \to 0} f(x + \Delta x) = f(x)$. Last equality we rewrite $\lim_{\Delta x \to 0} f(x + \Delta x) - f(x) = 0$. Here x is the fixed point, i.e. $f(x)$ is a constant as $\Delta x \to 0$, hence

$$\lim_{\Delta x \to 0} [f(x + \Delta x) - f(x)] = 0$$

Definition 8.3. The difference $f(x + \Delta x) - f(x)$ is called the increment of function $f(x)$ at a point x and denoted by Δy, i.e.

$$\Delta y = f(x + \Delta x) - f(x)$$

We have proved next theorem.

Theorem 8.1. (Necessary and sufficient condition for continuity of function) A function is continuous at the point x if and only if the limit of the increment of the function equals to zero as the increment of the independent variable approaches 0, i.e.

$$\lim_{\Delta x \to 0} \Delta y = 0 \tag{1.8}$$

Example 8.2. Prove that square function $y = x^2$ is continuous at each point in its domain $X = (-\infty; \infty)$.

Let us fix an arbitrary value of independent variable $x \in (-\infty; \infty)$ and Δx is an increment of this value. We have $f(x) = x^2$, $f(x+\Delta x) = (x+\Delta x)^2$ and the increment of the function

$$\Delta y = (x + \Delta x)^2 - x^2 = x^2 + 2x\Delta x + \Delta x^2 - x^2 = 2x\Delta x + \Delta x^2$$

The limit of this increment as $\Delta x \to 0$

$$\lim_{\Delta x \to 0}(2x\Delta x + \Delta x^2) = 2x \lim_{\Delta x \to 0} \Delta x + \lim_{\Delta x \to 0} \Delta x^2 = 2x \cdot 0 + 0 = 0$$

which means at each point in the domain of square function the necessary and sufficient condition of continuity (1.8) is satisfied, i.e. the square function is continuous.

Example 8.3. Prove that sine function $y = \sin x$ is continuous at each point in its domain $X = (-\infty; \infty)$.

Let us fix an arbitrary $x \in \mathbb{R}$, change this fixed value by Δx and find the corresponding increment of sine function

$$\begin{aligned} \Delta y &= \sin(x + \Delta x) - \sin x = 2 \sin \frac{x + \Delta x - x}{2} \cos \frac{x + \Delta x + x}{2} \\ &= 2 \sin \frac{\Delta x}{2} \cos\left(x + \frac{\Delta x}{2}\right) \end{aligned}$$

If $-\frac{\pi}{2} < x < \frac{\pi}{2}$ then $|\sin x| < |x|$. Therefore $\sin \frac{\Delta x}{2}$ is an infinitesimal as $\Delta x \to 0$. The function $\cos\left(x + \frac{\Delta x}{2}\right)$ is bounded. Consequently, the product of these is an infinitesimal, i.e.

$$\lim_{\Delta x \to 0} \Delta y = 0$$

that means the sine function is continuous in its domain.

1.2.10 Continuity of elementary functions

Using the condition (1.8) one can check the continuity of every basic elementary function (as we did it in case of square function and sine function). So we can prove the continuity of power functions, trigonometric functions and arc functions (inverse trigonometric functions), exponential and logarithmic functions in domains of these functions. Of course, every check demands some transformations and knowledge about basic elementary functions.

Theorem 9.1. If the functions $u = u(x)$ and $v = v(x)$ are continuous at a point x then

- the sum $u(x) + v(x)$ is continuous at x
- the difference $u(x) - v(x)$ is continuous at x
- the product $cu(x)$ where c is constant, is continuous at x
- the product $u(x)v(x)$ is continuous at x
- the quotient $\dfrac{u(x)}{v(x)}$ is continuous at x if $v(x) \neq 0$.
- (the continuity of composite function $y = f[\varphi(x)]$). If $u = \varphi(x)$ is continuous at x and $y = f(u)$ is continuous at corresponding u then composite function $f[\varphi(x)]$ is continuous at x.

Proof. Let us prove the first and the last assertion of this theorem. To prove the first we denote the sum $y = u(x) + v(x)$. We fix the point x in the common domain of $u(x)$ and $v(x)$ and increase this x by Δx. The corresponding increment of the sum

$$\Delta y = u(x + \Delta x) + v(x + \Delta x) - [u(x) + v(x)] = \Delta u + \Delta v$$

Using the continuity condition (1.8) of the functions u and v $\lim\limits_{\Delta x \to 0} \Delta u = 0$ and $\lim\limits_{\Delta x \to 0} \Delta v = 0$. According to the limit theorem 5.1

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} \Delta u + \lim_{\Delta x \to 0} \Delta v = 0$$

that is the necessary and sufficient condition for continuity of the sum at the point x is fulfilled.

To prove the last assertion we fix again a point x and change it by Δx. To the increment of the independent variable Δx there corresponds the increment of the function $u = \varphi(x)$

$$\Delta u = \varphi(x + \Delta x) - \varphi(x)$$

which is simultaneously an increment of the independent variable for the function $y = f(u)$. The corresponding increment of this function is

$$\Delta y = f(u + \Delta u) - f(u)$$

As assumed the function $u = \varphi(x)$ is continuous at x thus,

$$\lim_{\Delta x \to 0} \Delta u = 0$$

As well because of continuity of the function $y = f(u)$ at u

$$\lim_{\Delta u \to 0} \Delta y = 0$$

Therefore

$$\lim_{\Delta x \to 0} \Delta y = 0$$

that is the necessary and sufficient condition for the continuity of composite function at x is fulfilled.

Remark. In theorem 9.1 the fixed value of independent variable x can be an arbitrary value in the domain of sum, difference, etc. Consequently the sum, difference, product, quotient and composition of the continuous functions are continuous any time when these are defined.

In calculus, an elementary function is a function of one variable built from a finite number of basic elementary functions through composition and combinations using the four elementary operations (addition, subtraction, multiplication and division). Because of the continuity of basic elementary functions and theorem 9.1 we can formulate as a theorem an important conclusion.

Theorem 9.2. Every elementary function is continuous in its domain.

1.2.11 Points of discontinuity

Definition 10.1 The *point of discontinuity* of a function is the point at which the function is not continuous.

The definition of continuity of the function $f(x)$ at a point a yields that there are three possibilities for discontinuity:

- $f(a)$ does not exist
- $\lim_{x \to a} f(x)$ does not exist
- $\lim_{x \to a} f(x) \neq f(a)$

This subsection describes the classification of discontinuities. Consider a function $f(x)$, defined in a neighborhood of the point a at which $f(x)$ is discontinuous. Three situations can be distinguished.

Definition 10.2. It is said that the function $y = f(x)$ has *removable discontinuity* at a point a if $f(a)$ does not exist, but there $\exists \lim\limits_{x\to a} f(x) = b$.

This discontinuity can be removed to obtain a continuous function at a, or more precisely, the function

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \neq a \\ b & \text{if} \quad x = a \end{cases}$$

is continuous at a.

Example 10.1 The function $f(x) = \dfrac{\sin x}{x}$ has the removable discontinuity at the point $x = 0$ because $f(0)$ is not defined, but there exists $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ and the function $g(x)$, defined as

$$g(x) = \begin{cases} \frac{\sin x}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

is continuous at $x = 0$.

Definition 10.3 It is said that the function $y = f(x)$ has a *jump discontinuity or discontinuity of the first kind* at a point a if there exist finite one-sided limits

$$\lim_{x\to a-} f(x) = b_1$$

and

$$\lim_{x\to a+} f(x) = b_2$$

but $b_2 \neq b_1$.

Example 10.2. Using the example 3.1, one can say that the function $y = \dfrac{|x|}{x}$ has the jump discontinuity at $x = 0$ because

$$\lim_{x\to 0-} \frac{|x|}{x} = -1$$

and

$$\lim_{x\to 0+} \frac{|x|}{x} = 1$$

Definition 10.4. It is said that the function $y = f(x)$ has an *infinite discontinuity or discontinuity of the second kind* at a point a if at least on of the one-sided limits

$$\lim_{x\to a-} f(x) \bigvee \lim_{x\to a+} f(x)$$

is infinite or does not exist.

Example 10.3. The function $y = \dfrac{1}{x}$ has the infinite discontinuity at $x = 0$ because both one-sided limits are infinite

$$\lim_{x\to 0-} \frac{1}{x} = -\infty$$

and

$$\lim_{x\to 0+} \frac{1}{x} = \infty$$

Example 10.4. The function $y = \sin\dfrac{1}{x}$ has at the point $x = 0$ discontinuity of the second kind because there exists no

$$\lim_{x\to 0-} \sin\frac{1}{x}$$

neither

$$\lim_{x\to 0+} \sin\frac{1}{x}$$

1.2.12 Properties of continuous functions on closed interval

Let us consider a function $y = f(x)$ and a closed interval $[a;b]$ which is a subset of the domain of this function.

Theorem 11.1 (Extreme value theorem). A continuous on the closed interval $[a;b]$ function $f(x)$ has a maximum value and a minimum on this interval.

If M denotes the maximum value and m the minimum value of the function $f(x)$ on the closed interval $[a;b]$ then by theorem 11.1 there exists at least one point $\xi_1 \in [a;b]$ such that $f(\xi_1) = M$. Also there exists at least one point $\xi_2 \in [a;b]$ such that $f(\xi_2) = m$.

While the Extreme value theorem may seem intuitively obvious, it is a difficult theorem to prove.

Theorem 11.2. A continuous on the closed interval $[a;b]$ function $f(x)$ has each value between the minimum and maximum values.

If μ is some value between the minimum and maximum value, i.e. $m \leq \mu \leq M$ then there $\exists$ at least one point $\xi \in [a;b]$ such that $f(\xi) = \mu$.

Conclusion 11.3. If a function $f(x)$ has positive and negative values on the closed interval $[a;b]$ then this function has at least one zero on this interval (the equation $f(x) = 0$ has at least one root on the interval $[a;b]$).

Indeed, if the function has negative values on $[a;b]$ then the minimum value $m < 0$ and if it has positive values on $[a;b]$ then the maximum value

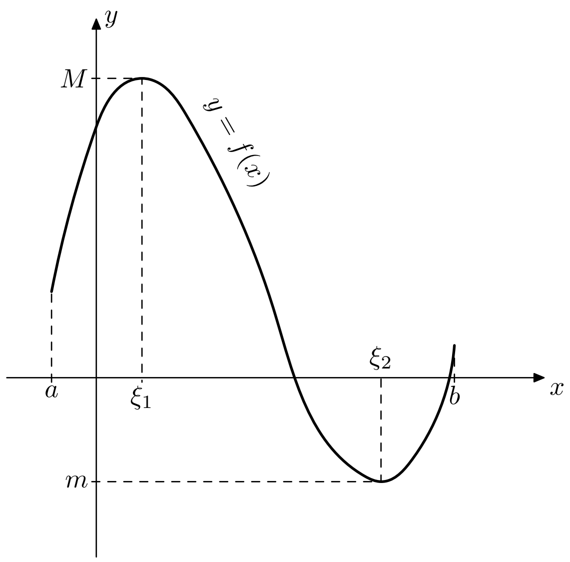

Figure 1.7: The maximum and minimum value of a continuous function

$M > 0$. Therefore $m < 0 < M$ and by theorem 11.2 there $\exists$ at least one $\xi \in [a; b]$ such that $f(\xi) = 0$.

Example 11.1. The equation

$$x^3 - 3x^2 + 2 = 0$$

has on closed interval $[0; 2]$ at least one root because $f(x) = x^3 - 3x^2 + 2$ is continuous on $(-\infty; \infty)$, hence on interval $[0; 2]$, $f(0) = 2$ and $f(2) = -2$.

Although in this context it is not important, one can check that this root is $x = 1$.

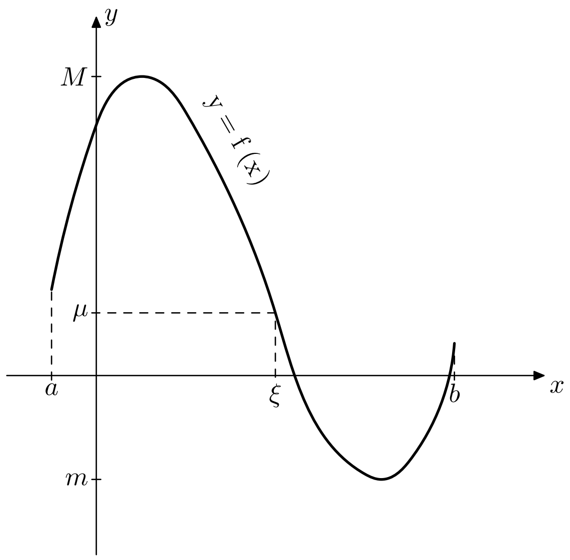

Figure 1.8: The value between the minimum and maximum values