2 Differentiation

2.1 Derivative of function

Let us fix an arbitrarily chosen point x in the domain of the function $y = f(x)$. Increasing this fixed value x by Δx we obtain the value of independent variable $x+\Delta x$. The value of function at this point is $f(x+\Delta x)$. The change of the function $\Delta y = f(x+\Delta x) - f(x)$ is called *the increment of the function* which corresponds to the increment of the independent variable Δx. The *average rate of change* of y with respect to x from x to $x + \Delta x$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Definition 1.1 The limit of the average rate of change of y as Δx approaches zero is called *the derivative* of $f(x)$ at x and denoted $f'(x)$.

According to this definition

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{2.1}$$

The derivative of the function is denoted $f'(x)$ or y'. These are Newton's notations of derivative. Also there is widely used Leibnitz's notation $\dfrac{dy}{dx}$ or $\dfrac{df}{dx}$.

Definition 2 The function which has derivative at x is called *differentiable* at x.

In other words, the function is differentiable at x if there exists the limit (2.1).

Consider the geometrical and physical concept to the derivative. For geometrical concept we use the graph of the function (Figure 2.1).

To the fixed value x there corresponds the point P on the graph of the function and to the variable value $x + \Delta x$ the point Q. Let us denote the angle of elevation of the secant line through points P and Q by φ. In the right triangle PRQ the angle at P is also φ (these are conjugate angles). The length of the opposite side RQ of this angle is Δy and the length of the adjacent side PR is Δx. Thus, the ratio of the increments of dependent variable and independent variable

$$\tan \varphi = \frac{\Delta y}{\Delta x}$$

equals to the slope of the secant line PQ. As $\Delta x \to 0$, i.e. $x + \Delta x \to x$, the point P remains fixed, Q moves along the curve toward P, and the secant

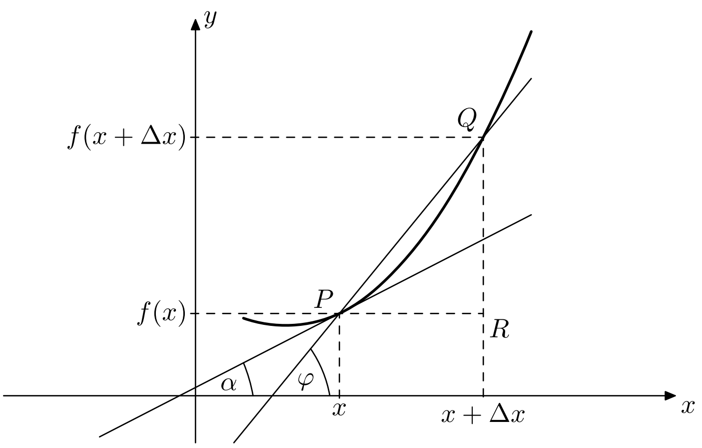

Figure 2.1: Geometrical concept of derivative

line through PQ changes its direction in such a way that its slope approaches the slope of the tangent line drawn to the graph at P. When α denotes the angle of elevation of tangent line at P then

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\varphi \to \alpha} \tan \varphi = \tan \alpha$$

Consequently, geometrically the derivative is the slope of the tangent line drawn to the graph of the function at the point with abscissa x.

If we treat the variable x as time variable, then the function $y = f(x)$ describes some mechanical or physical process, for example movement along straight line. At the fixed time x the moving particle is in the position $f(x)$ and at the time $x + \Delta x$ in the position $f(x + \Delta x)$. During time interval Δx the particle has passed the distance Δy. In this case the ratio $\dfrac{\Delta y}{\Delta x}$ is an average velocity of this particle during the time interval Δx. If Δx gets smaller, the average velocity gets closer to the velocity the particle has at a fixed time x. Thus the limit, as Δx approaches 0, i.e. the derivative of the movement is an instantaneous velocity at a time x.

When $f(x)$ describes the position of the moving particle then $f'(x)$ describes the instantaneous velocity of this moving particle.

More generally, if $f(x)$ describes some mechanical, physical etc. process then the derivative $f'(x)$ describes the instantaneous rate of change of this process.

2.2 Continuity and differentiability

The purpose of this subsection is to show that from differentiability of a function at a point it always follows continuity of the function at this point. The converse assertion is not true.

Theorem 2.1. If the function $y = f(x)$ is differentiable at x, then it is continuous at x.

Proof. Let the function $y = f(x)$ be differentiable at x, that means $\exists$ $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$. To obtain the necessary and sufficient condition of the continuity of the function holds, we find

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \Delta x = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \lim_{\Delta x \to 0} \Delta x = f'(x) \cdot 0 = 0,$$

what is we wanted to prove.

The following example proves that the converse assertion does not hold. The function $y = |x|$ is continuous at $x = 0$ but not differentiable at this point. The increment of the function at $x = 0$ is

$$\Delta y = |0 + \Delta x| - |0| = |\Delta x|$$

Hence

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} |\Delta x| = 0$$

i.e. there holds the necessary and sufficient condition of continuity at $x = 0$. From evaluation of one-sided limits at $x = 0$

$$\lim_{\Delta x \to 0-} \frac{|\Delta x|}{\Delta x} = -1$$

and

$$\lim_{\Delta x \to 0+} \frac{|\Delta x|}{\Delta x} = 1$$

it follows that there does not exist the limit $\lim\limits_{\Delta x \to 0} \dfrac{|\Delta x|}{\Delta x}$, that means the function $y = |x|$ does not have derivative at $x = 0$.

2.3 Derivatives of some basic elementary functions

In this subsection we find the derivatives of basic elementary functions using the definition of the derivative (2.1). Let us start with the constant function $y = c$. In this case $f(x) = c$ and for $\Delta x \neq 0$ $f(x + \Delta x) = c$ thus

$\Delta y = c - c = 0$. The derivative of a constant $c' = \lim\limits_{\Delta x \to 0} \dfrac{0}{\Delta x} = 0$. Here we have the first rule: the derivative of any constant equals to zero:

$$\boxed{c' = 0}$$

Second we find the derivative of a power function with an arbitrary natural exponent $y = x^n$. In this case $f(x) = x^n$, $f(x + \Delta x) = (x + \Delta x)^n$ and the increment of the function $\Delta y = (x + \Delta x)^n - x^n$.

By Newton's binomial formula

$$\Delta y = x^n + nx^{n-1}\Delta x + C_n^2 x^{n-2}\Delta x^2 + \ ... \ + \Delta x^n - x^n = \\ nx^{n-1}\Delta x + C_n^2 x^{n-2}\Delta x^2 + \ ... \ + \Delta x^n.$$

Dividing this equality by Δx we have

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + C_n^2 x^{n-2}\Delta x + \ ... \ + \Delta x^{n-1}$$

and the limit

$$(x^n)' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = nx^{n-1}$$

because all the terms starting from second contain some power of Δx with positive exponent. So, the derivative of the power function with natural exponent

$$\boxed{(x^n)' = nx^{n-1}} \tag{2.2}$$

Third we find the derivative of square root function $y = \sqrt{x}$. We find the increment of the function $\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$ and according to the definition of the derivative

$$(\sqrt{x})' = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\sqrt{x + \Delta x} - \sqrt{x})(\sqrt{x + \Delta x} + \sqrt{x})}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} = \\ \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}.$$

Fourth we find the derivative of reciprocal function $y = \dfrac{1}{x}$. The increment of the function is

$$\Delta y = \frac{1}{x + \Delta x} - \frac{1}{x} = \frac{x - x - \Delta x}{x(x + \Delta x)} = -\frac{\Delta x}{x(x + \Delta x)}$$

and due to the definition of the derivative

$$\left(\frac{1}{x}\right)' = \lim_{\Delta x \to 0} \frac{-1}{x(x + \Delta x)} = -\frac{1}{x^2} = -x^{-2}$$

The last two derivatives give an idea that the formula of the derivative of power function (2.2) holds not only for natural exponents but also for negative integer and fractional exponents. This fact we can prove later.

Fifth we find the derivative of sine function $y = \sin x$. We find the increment of the function $\Delta y = \sin(x+\Delta x) - \sin x = \sin x \cos \Delta x + \cos x \sin \Delta x - \sin x = \cos x \sin \Delta x - \sin x(1 - \cos \Delta x)$. By the definition of the derivative of the function (here we use some properties of the limits)

$$\begin{aligned}
(\sin x)' &= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x - \sin x(1 - \cos \Delta x)}{\Delta x} = \\
&= \lim_{\Delta x \to 0} \left(\frac{\cos x \sin \Delta x}{\Delta x} - \frac{\sin x(1 - \cos \Delta x)}{\Delta x} \right) = \\
&= \cos x \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} - \sin x \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = \\
&= \cos x - \sin x \lim_{\Delta x \to 0} \frac{(1 - \cos \Delta x)(1 + \cos \Delta x)}{\Delta x(1 + \cos \Delta x)} = \\
&= \cos x - \sin x \lim_{\Delta x \to 0} \frac{\sin^2 \Delta x}{\Delta x(1 + \cos \Delta x)} = \\
&= \cos x - \sin x \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \cdot \frac{\sin \Delta x}{1 + \cos \Delta x} = \cos x - \sin x \cdot 0 = \cos x.
\end{aligned}$$

Thus,

$$\boxed{(\sin x)' = \cos x}$$

As a result of similar transformations one can prove that

$$\boxed{(\cos x)' = -\sin x}$$

Seventh we find the derivative of the natural logarithm $y = \ln x$. We fix a value of x in the domain of this function $x > 0$ and find the increment of the function $\Delta y = \ln(x + \Delta x) - \ln x = \ln \dfrac{x + \Delta x}{x} = \ln \left(1 + \dfrac{\Delta x}{x}\right)$. By the definition of the derivative

$$(\ln x)' = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \ln \left(1 + \frac{\Delta x}{x}\right) = \lim_{\Delta x \to 0} \ln \left(1 + \frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}.$$

The value $x > 0$ is fixed and as $\Delta x \to 0$, $\dfrac{x}{\Delta x} \to \pm\infty$ and

$$\begin{aligned}
(\ln x)' &= \lim_{\frac{x}{\Delta x} \to \infty} \ln \left(1 + \frac{1}{\frac{x}{\Delta x}}\right)^{\frac{x}{\Delta x}\frac{1}{x}} = \\
&= \lim_{\frac{x}{\Delta x} \to \infty} \ln \left[\left(1 + \frac{1}{\frac{x}{\Delta x}}\right)^{\frac{x}{\Delta x}}\right]^{\frac{1}{x}} = \ln e^{\frac{1}{x}} = \frac{1}{x}.
\end{aligned}$$

Consequently

$$\boxed{(\ln x)' = \frac{1}{x}}$$

The derivatives of the rest basic elementary function we find in the following subsections.

2.4 Rules of differentiation

The finding a derivative of a function is called *differentiation* of given function. So, the rules of the differentiation are the rules of finding derivatives.

Let us have two functions $u = u(x)$ and $v = v(x)$. Assume that both of them are differentiable at x.

Theorem 4.1 (the sum rule). The derivative of the sum of the functions equals to the sum of derivatives of these functions:

$$[u(x) + v(x)]' = u'(x) + v'(x)$$

Proof. Denote the sum $y(x) = u(x) + v(x)$. The increment of this sum

$$\begin{aligned} \Delta y &= u(x + \Delta x) + v(x + \Delta x) - [u(x) + v(x)] = \\ &= u(x + \Delta x) - u(x) + v(x + \Delta x) - v(x) = \Delta u + \Delta v \end{aligned}$$

and with help the corresponding property of limits

$$y'(x) = \lim_{\Delta x \to 0} \frac{\Delta u + \Delta v}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = u'(x) + v'(x).$$

Example 4.1. Find the derivative of the function $y = x + \sin x$

By the sum rule $y' = (x + \sin x)' = x' + \sin x)' = 1 + \cos x$.

This rule also applies to additions of more than two functions

$$[u(x) + v(x) + w(x) + \ldots]' = u'(x) + v'(x) + w'(x) + \ldots$$

Example 4.2. Find the derivative of the function $y = x^3 + x^2 + x + 1$.

By the sum rule

$$y' = (x^3 + x^2 + x + 1)' = (x^3)' + (x^2)' + x' + 1' = 3x^2 + 2x + 1 + 0 = 3x^2 + 2x + 1$$

Theorem 4.2 (the product rule). The derivative of the product of two functions

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$$

Proof. Let us denote the product $y(x) = u(x)v(x)$. Then

$$\begin{aligned}\Delta y &= u(x+\Delta x)v(x+\Delta x) - u(x)v(x) = \\ &= u(x+\Delta x)v(x+\Delta x) - u(x)v(x+\Delta x) + u(x)v(x+\Delta x) - u(x)v(x) = \\ &= [u(x+\Delta x) - u(x)]v(x+\Delta x) + u(x)[v(x+\Delta x) - v(x)] = \\ &= \Delta u \cdot v(x+\Delta x) + u(x)\Delta v.\end{aligned}$$

By the properties of the limits

$$\begin{aligned}y'(x) &= \lim_{\Delta x\to 0} \frac{\Delta u \cdot v(x+\Delta x) + u(x)\Delta v}{\Delta x} = \\ &= \lim_{\Delta x\to 0} \frac{\Delta u}{\Delta x} \lim_{\Delta x\to 0} v(x+\Delta x) + u(x) \lim_{\Delta x\to 0} \frac{\Delta v}{\Delta x}.\end{aligned}$$

As we assumed, the function $v(x)$ is differentiable at x. The theorem 2.1 states that $v(x)$ is also continuous at x. According to the third condition of continuity $\lim\limits_{\Delta x\to 0} v(x+\Delta x) = v(x)$. It follows that $y'(x) = u'(x)v(x) + u(x)v'(x)$.

Example 4.3. Find the derivative of the function $y = x\sin x + \cos x$

$$\begin{aligned}y' &= (x\sin x + \cos x)' = (x\sin x)' + (\cos x)' = \\ &= x'\sin x + x(\sin x)' - \sin x = \sin x + x\cos x - \sin x = x\cos x.\end{aligned}$$

Conclusion 4.3 (the constant factor rule). The constant factor can be taken outside the derivative:

$$[c \cdot u(x)]' = c \cdot u'(x)$$

Indeed, by theorem 4.2 $[c\cdot u(x)]' = c' \cdot u(x) + c\cdot u'(x) = 0\cdot u(x) + c\cdot u'(x) = c \cdot u'(x)$.

Using this constant factor rule we find the derivative of the next basic elementary function $y = \log_a x$ ($a > 0$, $a \neq 1$). Here we use the change of base formula for logarithms $\log_a x = \dfrac{\ln x}{\ln a}$. We obtain

$$(\log_a x)' = \left(\frac{1}{\ln a}\ln x\right)' = \frac{1}{\ln a}(\ln x)' = \frac{1}{\ln a}\cdot\frac{1}{x} = \frac{1}{x\ln a}$$

So we have a new formula of differentiation

$$\boxed{(\log_a x)' = \frac{1}{x\ln a}}$$

Conclusion 4.4 (the subtraction rule). The derivative of the difference of two functions equals to the difference of the derivatives of these functions:

$$[u(x) - v(x)]' = u'(x) - v'(x).$$

To verify it we use the theorem 4.1 and conclusion 4.3: $[u(x) - v(x)]' = [u(x) + (-1)v(x)]' = u'(x) + [(-1)v(x)]' = u'(x) - v'(x)$.

Theorem 4.5 (the quotient rule). The derivative of the quotient

$$\left[\frac{u(x)}{v(x)}\right]' = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

provided $v(x) \neq 0$.

Proof. Denote the quotient

$$y(x) = \frac{u(x)}{v(x)}$$

and, as $v(x) \neq 0$, then $u(x) = y(x)v(x)$. By the product rule

$$u'(x) = y'(x)v(x) + y(x)v'(x)$$

hence

$$y'(x) = \frac{u'(x) - y(x)v'(x)}{v(x)}$$

Substituting $y(x)$ we obtain

$$y'(x) = \frac{u'(x) - \dfrac{u(x)v'(x)}{v(x)}}{v(x)} = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

which is we wanted to prove.

Using the theorem 4.5 we find the derivative of $y = \tan x$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Thus,

$$\boxed{(\tan x)' = \frac{1}{\cos^2 x}}$$

As well one can prove with help the theorem 4.5 that

$$\boxed{(\cot x)' = -\frac{1}{\sin^2 x}}$$

2.5 Derivative of inverse function

Here we restrict ourselves to one-to-one function $y = f(x)$, which has inverse function $x = f^{-1}(y)$.

Theorem 5.1. If the function $y = f(x)$ has a nonzero derivative at x $f'(x) \neq 0$, then the derivative of inverse function

$$(f^{-1}(y))' = \frac{1}{f'(x)}$$

Proof. The independent variable of the inverse function is y. By definition the derivative is

$$(f^{-1}(y))' = \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = \frac{1}{\lim\limits_{\Delta y \to 0} \dfrac{\Delta y}{\Delta x}}$$

According to the assumption $y = f(x)$ is differentiable at x, hence by the theorem 2.1 continuous at x. The inverse function $x = f^{-1}(y)$ of a continuous function is also continuous at y. Thus, $\Delta y \to 0$ yields $\Delta x \to 0$. Therefore

$$(f^{-1}(y))' = \frac{1}{\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}} = \frac{1}{f'(x)}$$

what is we wanted to prove.

We can read the assertion of the theorem 5.1 as follows: *the derivative of the inverse function equals to reciprocal of the derivative of given function.*

The opposite also holds. The derivative of the given function is the reciprocal of the derivative of the inverse function:

$$f'(x) = \frac{1}{(f^{-1}(y))'}. \tag{2.3}$$

The last equality we shall use to find some more derivatives of basic elementary functions. Let us start from exponential function $y = a^x$, ($a > 0$, $a \neq 1$). The inverse function of this function is $x = \log_a y$ and by (2.3)

$$(a^x)' = \frac{1}{(\log_a y)'} = \frac{1}{\dfrac{1}{y \ln a}} = y \ln a = a^x \ln a$$

The derivative of the exponential function

$$\boxed{(a^x)' = a^x \ln a}$$

In case of exponential function $y = e^x$ there holds $\ln e = 1$ and we obtain a special case of the last formula

$$\boxed{(e^x)' = e^x}$$

Next we find the derivative of $y = \arcsin x$. The inverse function is $x = \sin y$. The range of $y = \arcsin x$ and also the domain of inverse function is $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$. Therefore $\cos y \geq 0$ and by (2.3)

$$(\arcsin x)' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

Thus,

$$\boxed{(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}}$$

It is known that $\forall\ x \in [-1; 1]$ $\arcsin x + \arccos x = \frac{\pi}{2},$. It follows, that $\arccos x = \frac{\pi}{2} - \arcsin x$ and as $\frac{\pi}{2}$ is a constant, we have

$$\boxed{(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}}$$

Next we find the derivative of $y = \arctan x$. The inverse function is $x = \tan y$ and by (2.3)

$$(\arctan x)' = \frac{1}{(\tan y)'} = \frac{1}{\frac{1}{\cos^2 y}} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

Consequently

$$\boxed{(\arctan x)' = \frac{1}{1 + x^2}}$$

For each real x $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$, hence

$$\boxed{(\operatorname{arccot} x)' = -\frac{1}{1 + x^2}}$$

2.6 Derivative of the composite function

The components of the composite function $y = f[\varphi(x)]$ are $y = f(u)$ and $u = \varphi(x)$.

Theorem 6.1. If $u = \varphi(x)$ is differentiable at x and $y = f(u)$ is differentiable at u, then the composite function $y = f[\varphi(x)]$ is differentiable at x and

$$\{f[\varphi(x)]\}' = f'[\varphi(x)]\varphi'(x). \tag{2.4}$$

Proof. Denote the composite function $F(x) = f[\varphi(x)]$. Then $y = F(x)$ and the derivative

$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \\ &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}. \end{aligned}$$

The function $u = \varphi(x)$ is differentiable at x. By theorem 2.1 it is also continuous at x. It follows that there holds the necessary and sufficient condition of continuity, i.e. $\Delta x \to 0$ yields $\Delta u \to 0$. Therefore

$$F'(x) = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = f'(u)\varphi'(x)$$

which is we wanted to prove.

The equality (2.4) is also referred as *chain rule*. With help of this rule we find now the derivative of the common power function $y = x^\alpha$, where α is whatever real exponent and $x > 0$. We express the power function as

$$x^\alpha = e^{\alpha \ln x}$$

and by (2.4)

$$(x^\alpha)' = \left(e^{\alpha \ln x}\right)' = e^{\alpha \ln x}(\alpha \ln x)' = e^{\alpha \ln x} \cdot \frac{\alpha}{x} = x^\alpha \cdot \frac{\alpha}{x} = \alpha x^{\alpha - 1}$$

Thus, for every real α whenever x^α is defined

$$\boxed{(x^\alpha)' = \alpha x^{\alpha-1}}$$

Using

$$\left(e^{-x}\right)' = e^{-x}(-x)' = -e^{-x},$$

we have

$$(\sinh x)' = \left(\frac{1}{2}(e^x - e^{-x})\right)' = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

or

$$\boxed{(\sinh x)' = \cosh x}$$

In similar way one can prove

$$\boxed{(\cosh x)' = \sinh x}$$

According to the quotient rule we find

$$(\tanh x)' = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{(\sinh x)' \cosh x - \sinh x(\cosh x)'}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

Thus,

$$\boxed{(\tanh x)' = \frac{1}{\cosh^2 x}}$$

As well one can prove that

$$\boxed{(\coth x)' = -\frac{1}{\sinh^2 x}}$$

2.7 Implicit differentiation

A method called implicit differentiation makes use of the chain rule to differentiate implicitly defined functions $F(x, y) = 0$. One possibility for differentiation of implicit function is to write it in explicit form (if one can solve the equation for y) $y = f(x)$ and then differentiate it using the rules considered in previous subsections.

Often the implicit functions are multi-valued and then we need to differentiate every unique branch. In many cases the transformation of the function given implicitly to an explicit form can be complicated. Sometimes it is impossible to express y explicitly as a function of x and therefore y' cannot be found by explicit differentiation.

Consider some examples of implicit differentiation.

Example 7.1. Find y', if $x^2 + y^2 = r^2$.

We differentiate both sides of this equality with respect to x, taking into account that y^2 is a composite function: y is a function of x and the square function is the function of y. On the right side of the equality is the constant r^2. Using chain rule for y^2 we have the result of the differentiation

$$2x + 2y \cdot y' = 0$$

Solving this equation for y' we obtain

$$y' = -\frac{x}{y}$$

In order to differentiate function this explicitly, one have to solve the equation with respect to y $y = \pm\sqrt{r^2 - x^2}$. The derivative of the first unique branch $y = \sqrt{r^2 - x^2}$ is

$$y' = \frac{1}{2\sqrt{r^2 - x^2}} \cdot (-2x) = -\frac{x}{\sqrt{r^2 - x^2}}$$

The derivative of the second unique branch $y = -\sqrt{r^2 - x^2}$ is

$$y' = -\frac{1}{2\sqrt{r^2 - x^2}} \cdot (-2x) = -\frac{x}{-\sqrt{r^2 - x^2}}$$

In both cases the result matches with the result of the implicit differentiation.

Example 7.2. Find y', if $\sin(x + y) + \cos(xy) = 0$.

On the left side of this equality we have two composite functions. In the first the external function is sine function and internal function is $x + y$, in the second external function is cosine function and internal function xy. If we differentiate both sides of the equality with respect to x, we have

$$\cos(x + y) \cdot (1 + y') - \sin(xy) \cdot (y + xy') = 0$$

Removing parenthesis gives

$$\cos(x + y) + y' \cos(x + y) - y \sin(xy) - xy' \sin(xy) = 0$$

or

$$y' [\cos(x + y) - x \sin(xy)] = y \sin(xy) - \cos(x + y)$$

and

$$y' = \frac{y \sin(xy) - \cos(x + y)}{\cos(x + y) - x \sin(xy)}$$

2.8 Logarithmic differentiation

Logarithmic differentiation is a means of differentiating algebraically complicated functions or functions for which the ordinary rules of differentiation do not apply. First of all we have to use logarithmic differentiation if we have the function $y = [f(x)]^{g(x)}$, i.e. if we have variable base to variable exponent. In power function the exponent has to be constant and in exponential function the base has to be constant. So, we cannot treat this kind of function as power function (because the exponent $g(x)$ depends on x). Also we cannot treat this function as exponential function (because $f(x)$ depends on x).

Taking a logarithm of such function gives us possibility to transform the function so that the ordinary rules of differentiation will be applicable. If we take a logarithm to base e, we have

$$\ln y = \ln[f(x)]^{g(x)} = g(x) \ln f(x)$$

and $[f(x)]^{g(x)}$ has been transformed to product and we have the possibility to differentiate it using the product rule and the chain rule. The variable y is the function of the independent variable x, so the left side of the equality $\ln y$ is composite function and its derivative

$$(\ln y)' = \frac{1}{y} y'$$

Example 8.1. Find the derivative of $y = (x^2 + 1)^x$.

Taking a natural logarithm gives us

$$\ln y = x \ln(x^2 + 1)$$

and the differentiation of this equality

$$\frac{1}{y} y' = \ln(x^2 + 1) + x \frac{1}{x^2 + 1} 2x$$

or

$$\frac{1}{y} y' = \ln(x^2 + 1) + \frac{2x^2}{x^2 + 1}$$

Multiplying both sides of the last equality by y, we have

$$y' = y \left(\ln(x^2 + 1) + \frac{2x^2}{x^2 + 1} \right)$$

and after substituting y we obtain the desired derivative

$$y' = (x^2 + 1)^x \left(\ln(x^2 + 1) + \frac{2x^2}{x^2 + 1} \right)$$

Example 8.2. Find the derivative of the function $y = \dfrac{x^3 \sqrt{x - 1}}{\sqrt[5]{(x + 3)^2}}$.

The derivative of this function can be found without logarithmic differentiation. It is possible to differentiate this function by applying the quotient rule, the product rule and the chain rule. This would be quite complicated. We can simplify the differentiation by taking logarithms of both sides. We need to use the properties of logarithms to expand the right side as follows

$$\ln y = \ln \frac{x^3 \sqrt{x - 1}}{\sqrt[5]{(x + 3)^2}} = 3 \ln x + \frac{1}{2} \ln(x - 1) - \frac{2}{5} \ln(x + 3)$$

Now the differentiation process will be simpler:

$$\frac{1}{y}y' = 3 \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{x-1} - \frac{2}{5} \cdot \frac{1}{x+3}$$

Multiplying both sides of equality by y

$$y' = y\left[\frac{3}{x} + \frac{1}{2(x-1)} - \frac{2}{5(x+3)}\right]$$

and substituting y we have the result

$$y' = \frac{x^3\sqrt{x-1}}{\sqrt[5]{(x+3)^2}}\left[\frac{3}{x} + \frac{1}{2(x-1)} - \frac{2}{5(x+3)}\right]$$

2.9 Parametric differentiation

Let us consider the parametric representation of a function

$$\begin{cases} x = \varphi(t) \\ y = \psi(t). \end{cases}$$

Assume that both functions of the variable t are one-valued and differentiable and the derivative of x with respect to t $\frac{dx}{dt} \neq 0$ and $x = \varphi(t)$ has one-valued inverse function $t = \Phi(x)$.

The variable y is composite function with respect to x, i.e.

$$y = \psi[\Phi(x)]$$

and by chain rule

$$\frac{dy}{dx} = \psi'[\Phi(x)] \cdot \Phi'(x) \tag{2.5}$$

According to the rule of the derivative of the inverse function

$$\Phi'(x) = \frac{1}{\varphi'(t)} = \frac{1}{\frac{dx}{dt}}$$

Using notations $\psi'[\Phi(x)] = \psi'(t) = \frac{dy}{dt}$, we obtain from (2.5)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

In calculus the derivative by parameter is denoted

$$\frac{dx}{dt} = \dot{x}$$

and read "x-dot". As well

$$\frac{dy}{dt} = \dot{y}$$

which is read "y-dot". We can conclude that the derivative of a function given in parametric form is

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \tag{2.6}$$

Example 9.1. The parametric representation of the implicit function $x^2 + y^2 = r^2$ is

$$\begin{cases} x = r\cos t \\ y = r\sin t. \end{cases}$$

If we find the derivatives of both functions with respect to parameter t, we have $\dot{x} = -r\sin t$ and $\dot{y} = r\cos t$ and by the rule (2.6) we obtain the result

$$\frac{dy}{dx} = -\frac{r\cos t}{r\sin t} = -\frac{\cos t}{\sin t}$$

or

$$\frac{dy}{dx} = -\frac{x}{y}$$

which is the same we have got in Example 7.1.

Example 9.2. Find the slope of the tangent line of cycloid

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$$

at the point that corresponds to the value of parameter $t = \frac{\pi}{2}$.

The slope of the tangent line of the curve (the graph of the function) at a point equals to the value of derivative at this point. Therefore we need to evaluate the derivative at the point with $t = \frac{\pi}{2}$. To obtain that we find $\dot{x} = a(1 - \cos t)$ and $\dot{y} = a\sin t$, then by (2.6)

$$\frac{dy}{dx} = \frac{a\sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t}$$

The value of the derivative at the point where $t = \frac{\pi}{2}$, is

$$\frac{\sin\frac{\pi}{2}}{1 - \cos\frac{\pi}{2}} = 1$$

Consequently the slope of the tangent line at the point, where $t = \frac{\pi}{2}$, equals to 1.

2.10 Differential of function

In many cases it is sufficient to consider instead of the increment of a function its linear part (sometimes called principal part).

The derivative of the function $y = f(x)$ at the point x is defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

But then the variable $\frac{\Delta y}{\Delta x}$ is the sum

$$\frac{\Delta y}{\Delta x} = f'(x) + \alpha$$

where α is an infinitesimal as $\Delta x \to 0$. Multiplying both sides of the last equality by Δx, we have

$$\Delta y = f'(x)\Delta x + \alpha \Delta x \tag{2.7}$$

On the right side of equality (2.7) the first addend is for fixed value of x linear with respect to Δx, but the second addend is an infinitesimal of a higher order with respect to Δx because

$$\lim_{\Delta x \to 0} \frac{\alpha \Delta x}{\Delta x} = 0$$

Definition 1. The linear part $f'(x)\Delta x$ of the increment of the function (2.7) is called the *differential of the function* and denoted dy.

Due to this definition

$$dy = f'(x)\Delta x$$

If the independent and dependent variable are equal, i,e, $y = x$ then $y' = 1$ and $dy = dx = 1 \cdot \Delta x$. Consequently for independent variable x there holds $dx = \Delta x$, that means for the independent variable two concepts increment and differential coincide. Hence, the differential of the function $f(x)$ can be written

$$dy = f'(x)dx \tag{2.8}$$

Example 10.1. Find the expression of the differential of the function $y = \arctan \sqrt{x}$.

By the chain rule

$$y' = \frac{1}{1+x} \frac{1}{2\sqrt{x}}$$

and using (2.8) we write the expression of the differential

$$dy = \frac{1}{1+x} \frac{1}{2\sqrt{x}} dx = \frac{dx}{2(1+x)\sqrt{x}}$$

Example 10.2. Evaluate the increment and the differential of the function $y = x^2$ if the independent variable x increases from 1 to $1,05$.

First we find the increment of the function

$$\Delta y = 1,05^2 - 1^2 = 0,1025$$

The increment of the independent variable is $dx = \Delta x = 0,05$ the derivative $y' = 2x$ thus, the value of the differential $dy = 2 \cdot 1 \cdot 0,05 = 0,1$.

Next let us find out, what is the geometric interpretation of the differential of the function.

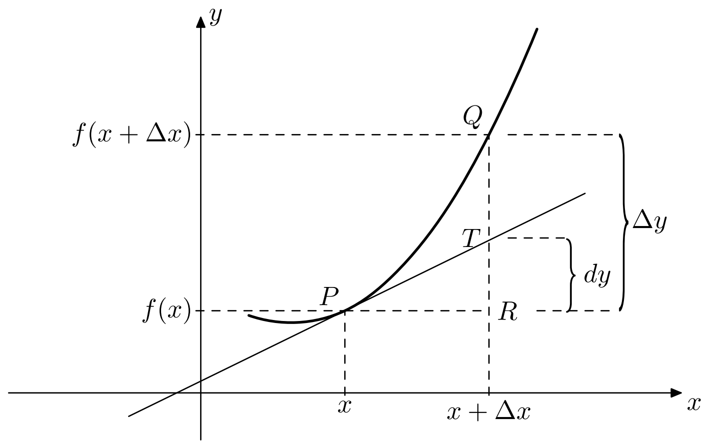

Figure 2.2: the differential of the function

The derivative of the function is the slope of the tangent line drawn to the graph at the point P (with abscissa x) or the tangent of the angle of elevation. The product $f'(x)dx$ equals to the length of the side RT of the right triangle PRT, i.e. the differential of the function is the length of the interval RT.

Consequently the differential of the function means the increment of y as x increases by Δx, if the movement along the curve has been replaced with the movement along the tangent line.

Mechanically the derivative means the instantaneous velocity and this velocity is a variable. If we fix a time x, we fix $f'(x)$, i.e. the instantaneous velocity of the moving object. If this moving object keeps on moving with this constant speed, the product $f'(x)dx$ equals to the distance this moving object has passed during the time interval Δx with that constant speed.

If Δx is sufficiently small then, since the difference of Δy and dy is a quantity, which is the infinitesimal of a higher order with respect to Δx, we may write $\Delta y \approx dy$. Using the definitions of the increment and the differential of the function, we get

$$f(x+\Delta x) - f(x) \approx f'(x)\Delta x$$

which yields the approximate formula

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x. \tag{2.9}$$

The formula (2.9) is applicable only for relatively small increments Δx of the argument.

Example 10.3. Using the formula (2.9) let us calculate the approximate value of $\ln 0,9$.

Here we choose $x = 1$, $\Delta x = -0,1$ and the function $f(x) = \ln x$. The value of the function at $x = 1$ equals to $f(1) = \ln 1 = 0$ and the value of the derivative $f'(x) = \frac{1}{x}$ at $x = 1$ equals to $f'(1) = 1$.

Therefore, by (2.9) we get the approximate value

$$\ln 0,9 \approx 0 + 1 \cdot (-0,1) = -0,1$$

which differs from the exact value less than $0,0054$.

2.11 Higher order derivatives

The derivative $f'(x)$ of the function $y = f(x)$ is the function of the independent variable x again and it is possible to differentiate this derivative.

Definition 11.1. The derivative of the derivative of the function $y = f(x)$ is called second derivative and denoted $f''(x)$, that is

$$f''(x) = [f'(x)]'$$

The second order derivative is denoted also y''. The Leibnitz notation is

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

which we read "d squared y dx squared" or $\frac{d^2f}{dx^2}$

Example 11.1. Let us find the second derivative of the function $y = e^{-x^2}$.

First using the chain rule we find the derivative $y' = e^{-x^2}(-2x) = -2xe^{-x^2}$ and next using the product rule and the chain rule we find the second derivative

$$y'' = -2e^{-x^2} - 2xe^{-x^2} \cdot (-2x) = 2e^{-x^2}(2x^2 - 1)$$

The third derivative of the function $y = f(x)$ is defined as the derivative of the second derivative, i.e.

$$f'''(x) = [f''(x)]'$$

or by Leibnitz notation

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

Definition 11.2. The nth order derivative $f^{(n)}(x)$ of the function $y = f(x)$ is defined as the derivative of the $n-1$-st order derivative:

$$f^{(n)}(x) = \left[f^{(n-1)}(x)\right]'$$

By Leibnitz notation

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$$

Example 11.2. Find the nth order derivative of the sine function $y = \sin x$.

To find the nth order derivative we use the mathematical induction. First we form the base of the induction finding the few first derivatives

$$\begin{aligned}
y' &= \cos x = \sin\left(x + \frac{\pi}{2}\right), \\
y'' &= -\sin x = \sin(x+\pi) = \sin\left(x + 2\cdot\frac{\pi}{2}\right), \\
y''' &= -\cos x = \sin\left(x + 3\cdot\frac{\pi}{2}\right), \\
y^{(4)} &= \sin x = \sin(x+2\pi) = \sin\left(x + 4\cdot\frac{\pi}{2}\right).
\end{aligned}$$

Now we are able to form the induction hypothesis $y^{(n)} = \sin\left(x + n\cdot\frac{\pi}{2}\right)$. To prove it we find the $n+1$-st order derivative

$$y^{(n+1)} = \cos\left(x + n\cdot\frac{\pi}{2}\right) = \sin\left(x + \frac{n\pi}{2} + \frac{\pi}{2}\right) = \sin\left(x + (n+1)\frac{\pi}{2}\right)$$

.

2.12 Equations of tangent and normal lines of curve

As in previous subsections the curve is a graph of a function.

We shall use the point-slope equation of the line. If the line passes the point $P_0(x_0; y_0)$ and has the slope k, then the equation of this line is

$$y - y_0 = k(x - x_0)$$

The ordinate of the point with abscissa x_0 on the graph of the function $y = f(x)$ is $f(x_0)$. The slope of the tangent line is $f'(x_0)$. Thus, according the point-slope equation we get the equation of the tangent line

$$y - f(x_0) = f'(x_0)(x - x_0). \tag{2.10}$$

Definition 12.1. The perpendicular of the tangent line of the curve at a point is called the *normal line* or *normal* of the curve at this point.

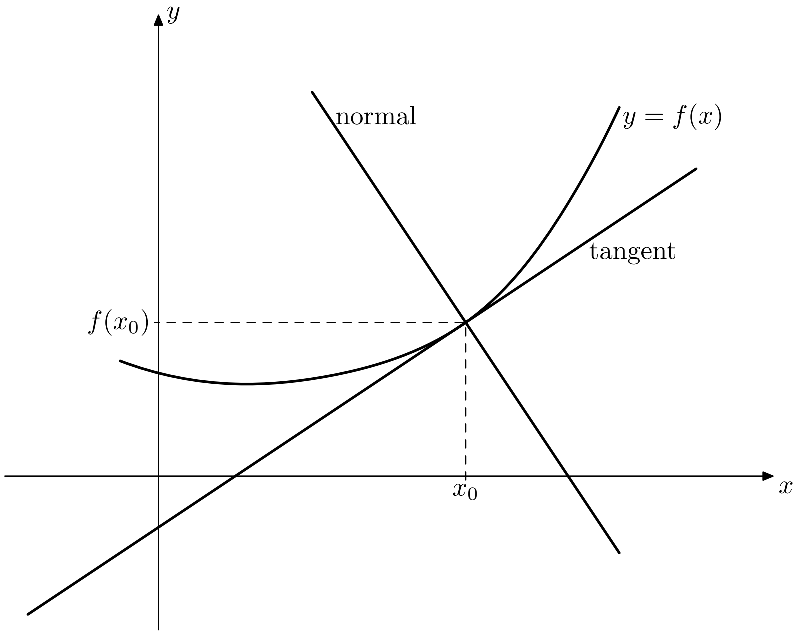

Figure 2.3: the tangent and normal line of the curve

If two lines are perpendicular and the slopes of the first and second lines are k_1 and k_2, respectively, then these slopes satisfy the condition $k_1 \cdot k_2 = -1$. Hence, if we know the slope k_1 of the given line, then the slope of the perpendicular line is $k_2 = -\dfrac{1}{k_1}$.

Therefore, the slope of the normal line at the point with abscissa x_0 is $-\dfrac{1}{f'(x_0)}$ and the normal has the equation

$$y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0). \tag{2.11}$$

Example 12.1 Find the equations of tangent and normal lines of the function $y = \cos x$ at the point with abscissa $x_0 = \frac{\pi}{6}$.

In our case the ordinate of the given point is $f(x_0) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. Using the derivative $f'(x) = -\sin x$, we find the the slope of the tangent line $f'\left(\frac{\pi}{6}\right) = -\frac{1}{2}$ and the slope of the normal line $-\frac{1}{f'(\frac{\pi}{6})} = 2$.

The tangent line has the equation

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{2}\left(x - \frac{\pi}{6}\right)$$

or

$$y = -\frac{1}{2}x + \frac{\pi + 6\sqrt{3}}{12}$$

The approximate value of the intercept of this line is $\frac{\pi + 6\sqrt{3}}{12} \approx 1,128$.

The normal has the equation

$$y - \frac{\sqrt{3}}{2} = 2\left(x - \frac{\pi}{6}\right)$$

or

$$y = 2x + \frac{3\sqrt{3} - 2\pi}{6}$$

The approximate value of the intercept of this line is $\frac{3\sqrt{3} - 2\pi}{6} \approx -0,181$.

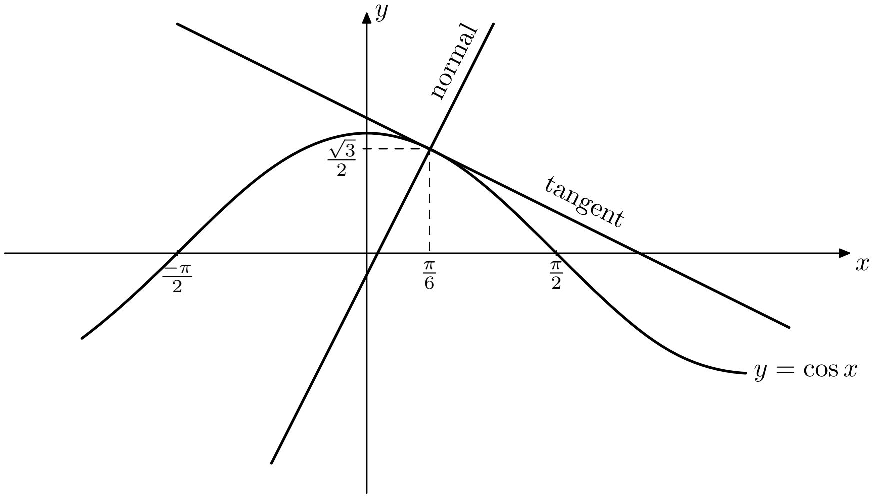

Figure 2.4: the tangent and the normal lines of the function $y = \cos x$ at the point $\left(\frac{\pi}{6}; \frac{\sqrt{3}}{2}\right)$