3 Applications of differentiation

Three theorems form a theoretical base of the applications of the differentiation: Rolle's, Cauchy and Lagrange theorems.

3.1 Rolle's theorem

Definition 1.1 A function $f(x)$ is said to have a *local maximum* at c if there is the neighborhood of c $(c-\varepsilon; c+\varepsilon)$ such that for each $x \in (c-\varepsilon; c+\varepsilon)$, $x \neq c$, there holds $f(x) < f(c)$.

Definition 1.2 A function $f(x)$ is said to have a *local minimum* at c if there is the neighborhood of c $(c-\varepsilon; c+\varepsilon)$ such that for each $x \in (c-\varepsilon; c+\varepsilon)$, $x \neq c$, there holds $f(x) > f(c)$.

A local maximum or local minimum of $f(x)$ is called a *local extremum* of $f(x)$.

First we prove one auxiliary theorem, so called Fermat's lemma.

Lemma 1.1 (Fermat's lemma). If the function $f(x)$ is differentiable at $c \in (a; b)$ and has local extremum at c, then $f'(c) = 0$.

Proof. Suppose the function $f(x)$ has a local maximum at $c \in (a; b)$. Then there exists a neighborhood $(c - \varepsilon; c + \varepsilon)$ such that for each $c + \Delta x \in (c - \varepsilon; c + \varepsilon)$, $\Delta x \neq 0$

$$f(c + \Delta x) < f(c)$$

or $f(c + \Delta x) - f(c) < 0$. If $\Delta x > 0$, then $\dfrac{f(c + \Delta x) - f(c)}{\Delta x} < 0$, hence

$$\lim_{\Delta x \to 0+} \frac{f(c + \Delta x) - f(c)}{\Delta x} \leq 0. \tag{3.1}$$

If $\Delta x < 0$, then $\dfrac{f(c + \Delta x) - f(c)}{\Delta x} > 0$, therefore

$$\lim_{\Delta x \to 0-} \frac{f(c + \Delta x) - f(c)}{\Delta x} \geq 0. \tag{3.2}$$

By assumption $f(x)$ is differentiable at c, i.e. there exists the limit

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

Consequently one-sided limits (3.1) and (3.2) are equal, which is possible only if both of them equal to zero. But then also $f'(c) = 0$.

If the function has local minimum at $c \in (a; b)$, the proof is analogous.

The point c such that $f'(c) = 0$ is called *stationary point* of the function.

Theorem 1.2 (Rolle's theorem). Suppose that a function $f(x)$ is continuous on a closed interval $[a;b]$, differentiable on the open interval $(a;b)$ and $f(a) = f(b)$. Then the function $f(x)$ has on the open interval $(a;b)$ at least one stationary point.

Proof. If we have a the constant function on $[a;b]$, then $f'(x) = 0$ for each $x \in (a;b)$, i.e. every point on $(a;b)$ is also the stationary point of $f(x)$.

Recall the theorem 11.1: a continuous on the closed interval $[a;b]$ function $f(x)$ has a maximum value and a minimum on this interval. If we have a non-constant function, then at least one of the maximum or minimum values has to differ from $f(a) = f(b)$. Suppose that the function has a maximum at $c \in (a;b)$. But then by Fermat lemma (as all the assumptions are satisfied) $f'(c) = 0$.

3.2 Cauchy theorem

In this subsection we shall formulate and prove the Cauchy mean value theorem.

Theorem 2.1 (Cauchy theorem). Suppose that two functions $f(x)$ and $g(x)$ are continuous on a closed interval $[a;b]$, differentiable on the open interval $(a;b)$ and $g'(x) \neq 0$ for all x in $(a;b)$. Then there exists at least one $c \in (a;b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \tag{3.3}$$

Proof. The assumptions yield that $g(a) \neq g(b)$ because otherwise the function $g(x)$ would satisfy the assumptions of Rolle's theorem. Then by Rolle's theorem there exists at least one $x \in (a;b)$ such that $g'(x) = 0$, which contradicts the assumption.

So $g(b) - g(a) \neq 0$ and we define a new function $F(x)$ on $[a;b]$ as follows:

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)]$$

Because of continuity of $f(x)$ and $g(x)$ on $[a;b]$ the function $F(x)$ is continuous on $[a;b]$ and from differentiability of $f(x)$ and $g(x)$ on $(a;b)$ it follows the differentiability of $F(x)$ on $(a;b)$. Moreover

$$F(a) = f(a) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(a) - g(a)] = 0$$

and

$$F(b) = f(b) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(b) - g(a)] = f(b) - f(a) - [f(b) - f(a)] = 0$$

that means the values of $F(x)$ at the end points are equal. Thus, the function $F(x)$ satisfies all the assumptions of the Rolle's theorem. By Rolle's theorem there exists at least one $c \in (a;b)$ such that $F'(c) = 0$. Then

$$0 = F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c)$$

or

$$f'(c) = \frac{f(b) - f(a)}{g(b) - g(a)} g'(c)$$

Dividing the last equality by $g'(c) \neq 0$, we complete the proof.

3.3 Lagrange theorem

Theorem 3.1 (Lagrange theorem). If a function $f(x)$ is continuous on the closed interval $[a;b]$ and differentiable on the open interval $(a;b)$, then there exists at least one $c \in (a;b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{3.4}$$

Proof. To prove Lagrange theorem it is sufficient to define in Cauchy theorem $g(x) = x$, because in this case $g(b) = b$, $g(a) = a$ and $g'(x) = 1$.

Often the assertion of Lagrange theorem is written as

$$f(b) - f(a) = f'(c)(b - a)$$

3.4 L'Hospital's rule

The L'Hospital's rule makes easier to evaluate the limits of quotients

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

if there are indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$. We have indeterminate form $\frac{0}{0}$ if $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$ and we have indeterminate form $\frac{\infty}{\infty}$ if $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = \pm\infty$. Application (or repeated application) of the rule often converts an indeterminate form to a determinate form, allowing easy evaluation of the limit. The differentiation of the numerator and denominator simplifies the quotient and/or converts it to a determinate form, allowing the limit to be evaluated more easily.

Theorem 4.1 (L'Hospital's rule $\frac{0}{0}$-form). Suppose $f(x)$ and $g(x)$ are differentiable in some neighborhood of a and $g'(x) \neq 0$ in this neighborhood (except possibly at a). Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and there exists the limit

$$\lim_{x\to a} \frac{f'(x)}{g'(x)}$$

then there exists also the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Proof. The functions $f(x)$ and $g(x)$ are differentiable in some neighborhood of a. From differentiability there follows the continuity in this neighborhood and because of third condition of continuity at a $f(a) = g(a) = 0$. If $x > a$, then by Cauchy theorem there exists $\xi \in (a; x)$ (if $x < a$, then $\xi \in (x; a)$) such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

If $x \to a$, then $a < \xi < x$ $(x < \xi < a)$ yields $\xi \to a$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{\xi\to a} \frac{f'(\xi)}{g'(\xi)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

.

Remark 4.2. If $\lim_{x\to a} f'(x) = \lim_{x\to a} g'(x) = 0$ and the functions $f'(x)$ and $g'(x)$ satisfy the assumptions of theorem 4.1 in some neighborhood of a, then we can apply L'Hospital's rule again:

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

Remark 4.3. Theorem 4.1 is valid for one-sided limits as well as the two-sided limit. This theorem is also true if $x \to \infty$ or $x \to -\infty$.

Theorem 4.4 (L'Hospital's rule $\frac{\infty}{\infty}$- form). If the functions $f(x)$ and $g(x)$ satisfy the assumptions of Cauchy theorem in some neighborhood of a $(a - \varepsilon; a + \varepsilon)$, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$ and there exists the limit

$$\lim_{x\to a} \frac{f'(x)}{g'(x)}$$

then there also exists the limit $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ and

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Because of the assumptions $\lim\limits_{x\to a}f(x)=\lim\limits_{x\to a}g(x)=\pm\infty$ the functions $f(x)$ and $g(x)$ satisfy the assumptions of Cauchy theorem in some neighborhood of a $(a-\varepsilon;a+\varepsilon)$, except at a.

Proof of Theorem 4.4 is omitted.

Example 4.1. Find the limit $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}$.

Here we have $\frac{0}{0}$-form. By L'Hospital's rule

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{(\ln(1+x))'}{x'}=\lim_{x\to 0}\frac{\frac{1}{1+x}}{1}=1$$

Example 4.2. Find the limit $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$.

Here we have also $\frac{0}{0}$-form. By L'Hospital's rule

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}=\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x}$$

Now we have $\frac{0}{0}$-form again. Applying the L'Hospital's rule second time, we obtain

$$...=\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$$

We have again $\frac{0}{0}$-form. Using the L'Hospital's rule third time, we have

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x}=2$$

Example 4.3. Find the limit $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\tan x}{\tan 3x}$.

In this limit we have $\frac{\infty}{\infty}$-form. By the L'Hospital's rule

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\tan 3x}=\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{\cos^2 x}}{\frac{3}{\cos^2 3x}}=\lim_{x\to\frac{\pi}{2}}\frac{\cos^2 3x}{3\cos^2 x}$$

We have got $\frac{0}{0}$-form. Applying the L'Hospital's rule two more times, we obtain

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\tan 3x} &= \lim_{x\to\frac{\pi}{2}}\frac{2\cos 3x(-\sin 3x)\cdot 3}{3\cdot 2\cos x(-\sin x)}=\lim_{x\to\frac{\pi}{2}}\frac{\sin 3x\cos 3x}{\sin x\cos x}=\\ &= \lim_{x\to\frac{\pi}{2}}\frac{\sin 3x}{\sin x}\lim_{x\to\frac{\pi}{2}}\frac{\cos 3x}{\cos x}=\frac{-1}{1}\lim_{x\to\frac{\pi}{2}}\frac{-3\sin 3x}{-\sin x}=-1\cdot\frac{3}{-1}=3.\end{aligned}$$

3.5 L'Hospital's rule for other indeterminate forms

Other indeterminate forms, such as $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞ or ∞^0 can sometimes be evaluated using l'Hospital's rule. To evaluate a limit involving one of these forms we have to convert given function to a quotient. In any of these cases we convert the limit to $\frac{0}{0}$-form or $\frac{\infty}{\infty}$-form.

The indeterminate form $0 \cdot \infty$ occurs in limits $\lim\limits_{x\to a} yz$, where $\lim\limits_{x\to a} y = 0$ and $\lim\limits_{x\to a} z = \infty$.

In this case we can write either $y \cdot z = \dfrac{y}{\frac{1}{z}}$ or $y \cdot z = \dfrac{z}{\frac{1}{y}}$. In the first case $\lim\limits_{x\to a} \dfrac{1}{z} = 0$ and in second case $\lim\limits_{x\to a} \dfrac{1}{y} = \infty$. Thus, the first transformation converts the limit $\lim\limits_{x\to a} yz$ to $\frac{0}{0}$-form and second transformation to $\frac{\infty}{\infty}$-form.

Example 5.1. Find the limit $\lim\limits_{x\to 0} x^n \ln x$, where $n \in \mathbb{N}$.

Obviously $\lim\limits_{x\to 0} x^n = 0$ and $\lim\limits_{x\to 0} \ln x = -\infty$. The limit converts to $\frac{\infty}{\infty}$-form if we write

$$\lim_{x\to 0} x^n \ln x = \lim_{x\to 0} \frac{\ln x}{x^{-n}}$$

By the L'Hospital's rule

$$\lim_{x\to 0} x^n \ln x = \lim_{x\to 0} \frac{\frac{1}{x}}{-nx^{-n-1}} = \lim_{x\to 0} \frac{x^n}{-n} = 0$$

The indeterminate form $\infty - \infty$ occurs in limits $\lim\limits_{x\to a}(y - z)$, where $\lim\limits_{x\to a} y = \infty$ and $\lim\limits_{x\to a} z = \infty$.

The expression $y - z$ can be transformed $y - z = \dfrac{1}{\frac{1}{y}} - \dfrac{1}{\frac{1}{z}} = \dfrac{\frac{1}{z} - \frac{1}{y}}{\frac{1}{y} \cdot \frac{1}{z}}$.

In this case $\lim\limits_{x\to a} \dfrac{1}{y} = 0$ and $\lim\limits_{x\to a} \dfrac{1}{z} = 0$, hence, the indeterminate form $\infty - \infty$ can be converted to $\dfrac{0}{0}$-form.

Example 5.2. Find the limit $\lim\limits_{x\to 1} \left(\dfrac{1}{\ln x} - \dfrac{1}{x-1} \right)$.

The limits of both denominators equal to zero, therefore, here we have the indeterminate form $\infty - \infty$. Writing these fractions to common denominator, we obtain the limit

$$\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \lim_{x\to 1} \frac{x - 1 - \ln x}{(x-1)\ln x}$$

which is $\frac{0}{0}$-form. By the L'Hospital's rule

$$\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}=\lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+(x-1)\frac{1}{x}}=\lim_{x\to 1}\frac{x-1}{x\ln x+x-1}$$

Now we have the limit with indeterminate form $\frac{0}{0}$. Applying once more the L'Hospital's rule, we have

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\frac{1}{\ln x+x\cdot\frac{1}{x}+1}=\frac{1}{2}$$

Considering indeterminate forms 0^0, 1^∞ or ∞^0, in all cases the limit can be written $\lim\limits_{x\to a} y^z$. In the first case $\lim\limits_{x\to a} y=0$ and $\lim\limits_{x\to a} z=0$, in the second case $\lim\limits_{x\to a} y=1$ and $\lim\limits_{x\to a} z=\infty$ and in the third case $\lim\limits_{x\to a} y=\infty$ and $\lim\limits_{x\to a} z=0$.

Using transformations $y^z=e^{\ln y^z}=e^{z\ln y}$ and the continuity of the function e^x, we can in any of three cases rewrite the limit

$$\lim_{x\to a} y^z=e^{\lim\limits_{x\to a} z\ln y}. \tag{3.5}$$

Now we have limit in exponent. In the first case $\lim\limits_{x\to a}\ln y=-\infty$, i.e. the transform produces the indeterminate form $0\cdot\infty$. In the second case $\lim\limits_{x\to a}\ln y=0$ and we have the indeterminate form $0\cdot\infty$ again. In the third case $\lim\limits_{x\to a}\ln y=\infty$. Thus, in any of three cases in the exponent of e there is indeterminate form $0\cdot\infty$.

Example 5.3. Find the limit $\lim\limits_{x\to 0}(\cot x)^{\sin x}$.

In this limit there is indeterminate form ∞^0. Using transformations

$$\lim_{x\to 0}(\cot x)^{\sin x}=e^{\lim\limits_{x\to 0}\sin x\ln\cot x}=e^{\lim\limits_{x\to 0}\frac{\ln\cot x}{\frac{1}{\sin x}}}$$

we obtain in the exponent of e the indeterminate form $\frac{\infty}{\infty}$. By the L'Hospital's rule

$$e^{\lim\limits_{x\to 0}\frac{\ln\cot x}{\frac{1}{\sin x}}}=e^{\lim\limits_{x\to 0}\frac{\frac{1}{\cot x}\left(-\frac{1}{\sin^2 x}\right)}{-\frac{\cos x}{\sin^2 x}}}=e^{\lim\limits_{x\to 0}\frac{\tan x}{\cos x}}=e^0$$

Hence, $\lim\limits_{x\to 0}(\cot x)^{\sin x}=1$.

Example 5.4. Applying the L'Hospital's rule let us prove that $\lim\limits_{|x|\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$.

As we know, in this limit we have the indeterminate form 1^∞.

We obtain the proof using transformations

$$\lim_{|x|\to\infty}\left(1+\frac{1}{x}\right)^x = e^{\lim\limits_{|x|\to\infty} x\ln\left(1+\frac{1}{x}\right)} = e^{\lim\limits_{|x|\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}}$$

and the L'Hospital's rule

$$e^{\lim\limits_{|x|\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}} = e^{\lim\limits_{|x|\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\left(\frac{1}{x}\right)'}{\left(\frac{1}{x}\right)'}}x = e^{\lim\limits_{|x|\to\infty}\frac{1}{1+\frac{1}{x}}} = e^1 = e$$

3.6 Taylor's formula

If $f(x)$ is a function having sufficiently higher order derivatives at the point $x = a$, then Taylors formula provides a representation of $f(x)$ as a polynomial with respect to powers of $x - a$.

Let us substitute in the approximate formula

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$

the fixed point instead of x by a and the variable point instead of $a + \Delta x$ by x, i.e. $x = a + \Delta x$. Then $\Delta x = x - a$ and in the approximate formula

$$f(x) \approx f(a) + f'(a)(x - a)$$

the polynomial

$$P_1(x) = f(a) + f'(a)(x - a)$$

is the linear polynomial with respect to $x - a$. The graph of this polynomial is the tangent line of $f(x)$ at a.

The purpose of the Taylor's formula is to get more exactness, adding to the first power of $x - a$ the terms containing the second, third etc. powers.

Thus, the purpose is to represent the function $f(x)$ in the neighborhood of a by the polynomial

$$P_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + ... + c_n(x - a)^n \quad (3.6)$$

with sufficient exactness.

Let us assume that $f(x)$ has the continuous derivatives up to $n+1$-st order in some neighborhood of a and the polynomial $P_n(x)$ satisfies the conditions at a as follows

$$\begin{aligned} P_n(a) &= f(a), \\ P'_n(a) &= f'(a), \\ P''_n(a) &= f''(a), \\ P'''_n(a) &= f'''(a), \\ &\ldots\ldots\ldots\ldots \\ P^{(n)}_n(a) &= f^{(n)}(a). \end{aligned} \tag{3.7}$$

Using to the conditions (3.7), we find the coefficients of the polynomial (3.6) via the values of the derivatives of the function $f(x)$ at a.

First $P_n(a) = c_0$ and the first condition in (3.7) gives

$$c_0 = f(a)$$

Differentiating the polynomial (3.6), we get

$$P'_n(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + ... + nc_n(x-a)^{n-1}$$

Therefore,t $P'_n(a) = c_1$ and the second condition in (3.7) gives

$$c_1 = f'(a) = \frac{f'(a)}{1!}$$

Differentiating the polynomial (3.6) second time, we obtain

$$P''_n(x) = 2c_2 + 6c_3(x-a) + ... + n(n-1)c_n(x-a)^{n-2}$$

Hence, $P''_n(a) = 2c_2$ and the third condition in (3.7) gives

$$c_2 = \frac{f''(a)}{2} = \frac{f''(a)}{2!}$$

Differentiating the polynomial (3.6) third time, we have

$$P'''_n(x) = 6c_3 + ... + n(n-1)(n-2)c_n(x-a)^{n-3}$$

Thus, $P'''_n(a) = 6c_3$ and the fourth condition in (3.7) yields

$$c_3 = \frac{f'''(a)}{6} = \frac{f'''(a)}{3!}$$

Differentiating the polynomial (3.6) nth time, we get the constant

$$P^{(n)}_n(x) = n(n-1)(n-2) \cdot ... \cdot 3 \cdot 2c_n$$

hence, $P_n^{(n)}(a) = n(n-1)(n-2) \cdot ... \cdot 3 \cdot 2c_n$ and the last condition in (3.7) gives

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Consequently, the polynomial satisfying the conditions (3.7) is (3.6)

$$P_n(x) = f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+...+\frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{3.8}$$

This polynomial is called *Taylor's polynomial* of degree n generated by $f(x)$ at the point a or Taylor's polynomial of degree n of the function $f(x)$ in powers $x-a$. The coefficients of the powers of $x-a$

$$\frac{f^{(k)}(a)}{k!} \quad (k = 0,\ 1,\ 2,\ \ldots,\ n)$$

are called the *Taylor's coefficients* of the function $f(x)$.

Example 6.1. Evaluate $\sqrt{1,2}$ using the Taylor's polynomials of the first, second, third and fourth degree. (By calculator $\sqrt{1,2} = 1,095445115$).

Here $f(x) = \sqrt{x}$, $a = 1$, $x = 1,2$ and $x-a = 0,2$. We find $f(1) = \sqrt{1} = 1$, the derivative $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ and its value $f'(1) = \frac{1}{2}$, the second derivative $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$ and its value $f''(1) = -\frac{1}{4}$, the third derivative $f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$ and its value $f'''(1) = \frac{3}{8}$ and the fourth derivative $f^{(4)}(x) = -\frac{15}{16}x^{-\frac{7}{2}}$ and its value $f^{(4)}(1) = -\frac{15}{16}$.

Using the Taylor's polynomial of first degree, we evaluate

$$f(a) + \frac{f'(a)}{1!}(x-a) = 1 + \frac{1}{2} \cdot 0,2 = 1,1$$

Using the Taylor's polynomial of second degree, we evaluate

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 = 1 + \frac{1}{2} \cdot 0,2 - \frac{\frac{1}{4}}{2!} \cdot 0,2^2 = 1,095$$

Using the Taylor's polynomial of third degree, we evaluate

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 =$$
$$1 + \frac{1}{2} \cdot 0,2 - \frac{\frac{1}{4}}{2!} \cdot 0,2^2 + \frac{\frac{3}{8}}{3!} \cdot 0.2^3 = 1,0955.$$

Using the Taylor's polynomial of fourth degree, we evaluate

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 =$$

$$1 + \frac{1}{2}\cdot 0,2 - \frac{\frac{1}{4}}{2!}\cdot 0,2^2 + \frac{\frac{3}{8}}{3!}\cdot 0.2^3 - \frac{\frac{15}{16}}{4!}\cdot 0,2^4 = 1,0954375.$$

From these calculations it yields: the using of the Taylor's polynomial of higher degree gives the higher degree of accuracy. But in general case we never attain an absolute accuracy. The value of a function and the value of the Taylor's polynomial differ by a quantity. Let us denote the error in the approximation of a function $f(x)$ by its Taylor polynomial $P_n(x)$ by

$$R_n(x) = f(x) - P_n(x)$$

This equality yields the formula

$$f(x) = P_n(x) + R_n(x) \tag{3.9}$$

which is called the *Taylor's formula* of the function $f(x)$ and $R_n(x)$ is called the *remainder of the nth degree of the Taylor's formula.* It is possible to prove that the remainder of the Taylor's formula is expressible as

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}[a + \Theta(x-a)], \tag{3.10}$$

where $0 < \Theta < 1$, i.e. $a + \Theta(x-a)$ is some point between a and x. This is called the *Lagrange form* of the remainder. (In calculus there are other forms of the Taylor's formula remainder, but we restrict ourselves with Lagrange form).

The absolute value of the Taylor's formula remainder $|R_n(x)| = |f(x) - P_n(x)|$ shows us how big is the difference of $f(x)$ and $P_n(x)$, that means how big is the error if we use the Taylor's polynomial (3.8) to evaluate the value of the function.

In the expression of the remainder we don't know the value Θ. All we know $0 < \Theta < 1$. This is the reason because the remainder cannot be evaluated but only estimated.

Example 6.2. Estimate the error which has been made evaluating the value of $\sqrt{1,2}$ using Taylor's polynomial of third degree.

To do it, we have to estimate the value $|R_3(x)|$. In Example 6.1 we got using Taylor's polynomial of the third degree $\sqrt{1,2} \approx 1,0955$. In Example 6.1 we also found the fourth derivative, thus, we have the expression of the remainder of the third degree (3.10)

$$R_3(x) = -\frac{(x-1)^4}{4!} \cdot \frac{15}{16\sqrt{(1+\Theta(x-1))^7}}$$

Now we estimate the remainder for $x = 1,2$

$$|R_3(1,2)| = \left|\frac{0,2^4}{24} \cdot \frac{15}{16\sqrt{(1+0,2\Theta)^7}}\right|$$

where $0 < \Theta < 1$. The positive fraction is the greatest if the denominator is the least. The denominator is the least if $1 + 0,2\Theta$ is the least and for each $0 < \Theta < 1$ there holds $1 + 0,2\Theta > 1$. Hence,

$$|R_3(1,2)| < \frac{0,2^4}{24} \cdot \frac{15}{16} = 0,0000625$$

that means the error that has been made, evaluating $\sqrt{1,2}$ using the Taylor's polynomial of third degree, does not exceed 0,0000625.

3.7 Maclaurin's polynomials of e^x, $\sin x$ and $\cos x$

The Taylor's formula at $a = 0$ or the Taylor's formula in powers x is called *Maclaurin's formula.* That means the Maclaurin's formula is a special case of Taylor's formula if $a = 0$. If we substitute in Taylor's formula $a = 0$, we obtain from (3.9) the Maclaurin's formula

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x), \quad (3.11)$$

and from the remainder of the Taylor's formula (3.10) we get the remainder of the Maclaurin's formula

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\Theta x) \quad (3.12)$$

where $0 < \Theta < 1$.

The polynomial in (3.11)

$$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n \quad (3.13)$$

is called *Maclaurin's polynomial.* In following we find the Maclaurin's formula of the functions e^x, $\sin x$ and $\cos x$.

3.7.1 Maclaurin's formula of e^x

Let $f(x) = e^x$ and find $f(0) = 1$, the derivative $f'(x) = e^x$ and its value at $x = 0$ $f'(0) = 1$, ..., the nth derivative $f^{(n)}(x) = e^x$ and its value at $x = 0$ $f^{(n)}(0) = 1$. Hence, by (3.11) the Maclaurin's formula of e^x (3.11) is

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{n!}x^n + R_n(x)$$

or

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + R_n(x)$$

which has by (3.12) the reminder

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} e^{\Theta x}$$

where $0 < \Theta < 1$. Let us prove that for each $x \in \mathbb{R}$ the limit

$$\lim_{n\to\infty} R_n(x) = 0 \tag{3.14}$$

For the fixed value of x is $e^{\Theta x}$ is bounded,

$$\frac{x^{n+1}}{(n+1)!} = \frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \cdot \ldots \cdot \frac{x}{n+1}.$$

Let q be a real number, satisfying the condition $0 < q < 1$. For each of such value of q there exists $N > 0$ such that if $k > N$, then $\frac{|x|}{k} < q$. Therefore,

$$\left|\frac{x^{n+1}}{(n+1)!}\right| = \left|\frac{x}{1}\right| \left|\frac{x}{2}\right| \ldots \left|\frac{x}{N}\right| \left|\frac{x}{N+1}\right| \ldots \left|\frac{x}{n+1}\right| <$$

$$< \left|\frac{x}{1}\right| \left|\frac{x}{2}\right| \ldots \left|\frac{x}{N}\right| q \cdot q \cdot \ldots q,$$

in which the factor q is repeated $n + 1 - N$ times. Hence,

$$\left|\frac{x^{n+1}}{(n+1)!}\right| < \left|\frac{x}{1}\right| \left|\frac{x}{2}\right| \ldots \left|\frac{x}{N}\right| \cdot q^{n+1-N}$$

The condition $0 < q < 1$ yields

$$\lim_{n\to\infty} q^{n+1-N} = q^{1-N} \lim_{n\to\infty} q^n = 0$$

Consequently $\frac{x^{n+1}}{(n+1)!}$ is an infinitesimal as $n \to \infty$ and $R_n(x)$ is the product of a bounded quantity and an infinitesimal, which is infinitesimal, that means

$$\lim_{n\to\infty} R_n(x) = 0$$

The condition (3.14) for every fixed value of $x \in \mathbb{R}$ says that using Maclaurin's polynomial we can evaluate e^x with any degree of accuracy if we take in the polynomial

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

sufficient number of terms, i.e. if we take n sufficiently large.

3.7.2 Maclaurin's formula of $\sin x$

In subsection 11.2 we have found that the nth order derivative of $f(x) = \sin x$ is $f^{(n)}(x) = \sin\left(x + n\frac{\pi}{2}\right)$. Evaluate $f(0) = 0$, $f'(0) = \sin\frac{\pi}{2} = 1$, $f''(0) = \sin\pi = 0$, $f'''(0) = \sin\frac{3\pi}{2} = -1$, $f^{(4)}(0) = \sin 2\pi = 0$, $f^{(5)}(0) = \sin\frac{5\pi}{2} = 1$ etc. Hence, all the derivatives of even order the function $\sin x$ at 0 equal to 0, the odd order derivatives $f^{(2n+1)}(0) = 1$ if n is even and $f^{(2n+1)}(0) = -1$ if n is odd.

Thus, the Maclaurin's formula (3.11) of $\sin x$ is

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n+1}(x)$$

whose remainder is by (3.12)

$$R_{2n+1}(x) = \frac{x^{2n+2}}{(2n+2)!}\sin\left(\Theta x + (2n+2)\frac{\pi}{2}\right) = \frac{x^{2n+2}}{(2n+2)!}\sin\left(\Theta x + (n+1)\pi\right),$$

where $0 < \Theta < 1$. It is possible to prove again that for each fixed value of $x \in \mathbb{R}$ the limit of the remainder

$$\lim_{n\to\infty} R_{2n+1}(x) = 0$$

i.e. using Maclaurin's polynomial, it is possible to evaluate the values of $\sin x$ with any degree of accuracy if we take in Maclaurin's polynomial sufficient number of terms.

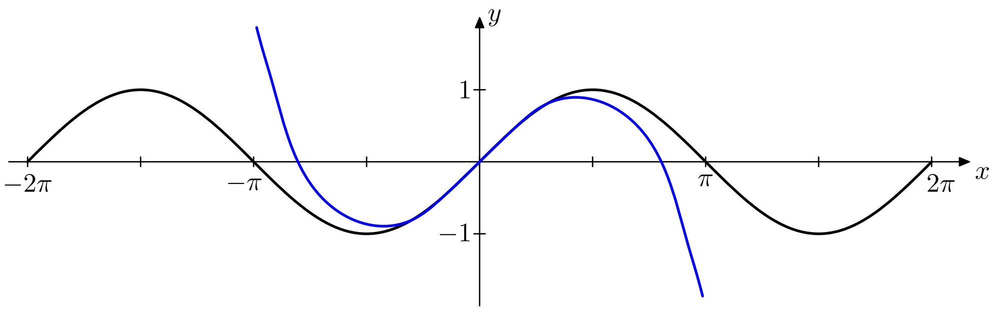

Figure 3.1: Sine function and the Maclaurin's polynomial of third degree

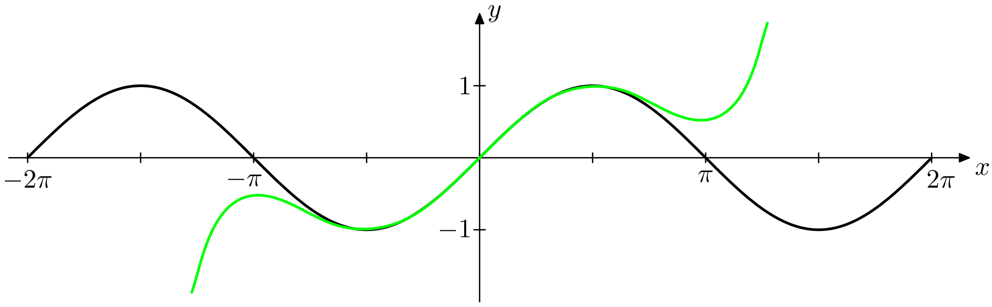

Figure 3.2: Sine function and the Maclaurin's polynomial of fifth degree

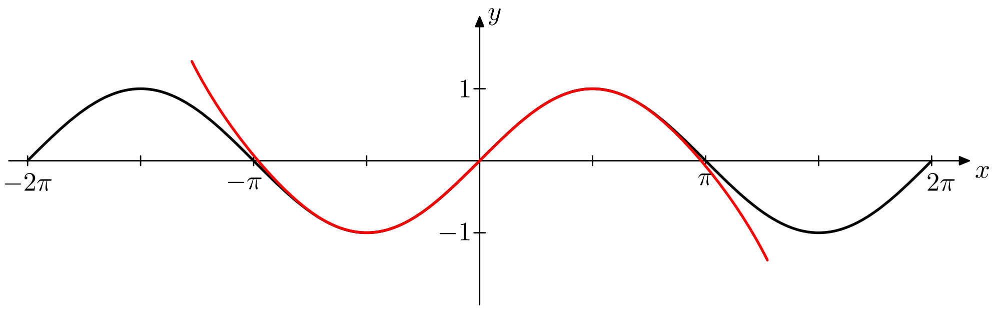

Figure 3.3: Sine function and the Maclaurin's polynomial of seventh degree

3.7.3 Maclaurin's formula of $\cos x$

The nth order derivative of function $f(x) = \cos x$ is $f^{(n)}(x) = \cos\left(x + n \cdot \frac{\pi}{2}\right)$. Evaluate $f(0) = 1$, $f'(0) = \cos\frac{\pi}{2} = 0$, $f''(0) = \cos\pi = -1$, $f'''(0) = \cos\frac{3\pi}{2} = 0$, $f^{(4)}(0) = \cos 2\pi = 1$ etc. All the derivatives of odd order of $\cos x$ at 0 equal to 0. The derivatives of even order $f^{(2n)}(0) = 1$ if n is even number and $f^{(2n)}(0) = -1$ if n is odd number. Thus, the Maclaurin's formula (3.11) of $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n}(x)$$

whose remainder is by (3.12)

$$R_{2n}(x) = \frac{x^{2n+1}}{(2n+1)!} \cos\left(\Theta x + (2n+1)\frac{\pi}{2}\right),$$

where $0 < \Theta < 1$. Here also for each fixed value of $x \in \mathbb{R}$ the limit of the remainder

$$\lim_{n\to\infty} R_{2n}(x) = 0$$

which means that the value of $\cos x$ can be evaluated with any degree of accuracy if we take in the polynomial

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!}$$

n sufficiently large.

Example 7.1. Evaluate $\cos 0,5$ by Maclaurin's polynomial, taking $n = 2$ and, using remainder, estimate the maximal error ($0,5 = 28°38'52''$).

First evaluate

$$\cos 0,5 \approx 1 - \frac{0,5^2}{2!} + \frac{0,5^4}{4!} = 1 - 0,125 + +0,0026 = 0,8776$$

and using remainder

$$R_4(x) = \frac{x^5}{5!} \cos\left(\Theta x + \frac{5\pi}{2}\right)$$

where $0 < \Theta < 1$, estimate the error. As for every $x \in \mathbb{R}$ holds $|\cos x| \leq 1$, then

$$|R_4(0,5)| = \left|\frac{0,5^5}{5!}\right| \left|\cos\left(0,5\Theta + \frac{5\pi}{2}\right)\right| \leq \frac{0,5^5}{5!} = \frac{0,03125}{120} = 0,000261$$

3.8 Increase and decrease of function

Theorem 8.1. If the function $f(x)$ is increasing and differentiable in the interval $(a;b)$, then $f'(x) \geq 0$ in this interval.

Proof. Let us fix $x \in (a;b)$ and choose $\Delta x > 0$ such that $x + \Delta x \in (a;b)$. Then $x + \Delta x > x$. By the assumption $f(x + \Delta x) > f(x)$, which implies $\Delta y > 0$ and $\dfrac{\Delta y}{\Delta x} > 0$. If $\Delta x < 0$, then $x + \Delta x < x$ and according to the assumption $f(x + \Delta x) < f(x)$, i.e. $\Delta y < 0$ and $\dfrac{\Delta y}{\Delta x} > 0$. By limit theorem

$$f'(x) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} \geq 0$$

Theorem 8.2. If the function $f(x)$ is decreasing and differentiable in the interval $(a;b)$, then $f'(x) \leq 0$ in this interval.

The proof of this theorem is analogous to the proof of theorem 8.1..

Theorem 8.3. If the function $f(x)$ is differentiable in the interval $(a;b)$ and $f'(x) > 0$ in this interval, then the function $f(x)$ is increasing in this interval.

Proof. Let us fix two values of argument x_1 and x_2 in the interval $(a;b)$ such that $x_1 < x_2$. Differentiability of the function in $(a;b)$ implies the differentiability in $(x_1;x_2)$. In this interval we can apply Lagrange theorem: there exists $c \in (x_1;x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

As assumed $f'(c) > 0$ and the choice of x_1 and x_2 implies $x_2 - x_1 > 0$. The product of two positive quantities is positive, thus, $f(x_2) - f(x_1) > 0$ or $f(x_2) > f(x_1)$ which means the function is increasing.

In the same manner we can prove the following theorem.

Theorem 8.4. If the function $f(x)$ is differentiable in the interval $(a;b)$ and $f'(x) < 0$ in this interval, then the function $f(x)$ is decreasing in this interval.

The theorems 8.3 and 8.4 tell us how the derivative can be used to check whether a function is increasing or decreasing on an interval. We shall denote the intervals of increase by $X \uparrow$ and intervals of decrease by $X \downarrow$

Example 8.1 Find the intervals of increase and decrease of the function $y = x^2e^{-x}$.

The domain of this function $X = \mathbb{R}$. Find the derivative $y' = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2-x)$. By theorem 8.3 we determine the intervals of increase, solving the inequality $xe^{-x}(2-x) > 0$ and by theorem 8.4 the intervals of decrease, solving the opposite inequality $xe^{-x}(2-x) < 0$. For each $x \in \mathbb{R}$ the exponential function $e^{-x} > 0$. Hence, the first inequality is equivalent to $x(2-x) > 0$ and the second inequality to $x(2-x) < 0$. The solution of the first inequality gives us the interval of increase of given function $X \uparrow = (0;2)$ And the solution of the second inequality gives the intervals of decrease of this function $X \downarrow = (-\infty;0)$ and $X \downarrow = (2;\infty)$.

3.9 Local extrema of function

Definition 9.1. It is said that the function $f(x)$ has local maximum at x_1 if x_1 has a neighborhood $(x_1-\varepsilon;x_1+\varepsilon)$ such that for each $x \in (x_1-\varepsilon;x_1+\varepsilon)$ there holds $f(x) < f(x_1)$.

Definition 9.2. It is said that the function $f(x)$ has local minimum at x_2 if x_2 has a neighborhood $(x_2-\varepsilon;x_2+\varepsilon)$ such that for each $x \in (x_2-\varepsilon;x_2+\varepsilon)$ there holds $f(x) > f(x_2)$.

Denoting $x = x_1 + \Delta x$, we can rewrite the condition $f(x) < f(x_1)$ as $f(x_1 + \Delta x) - f(x_1) < 0$ or $\Delta y < 0$.

Conclusion 9.1. The function has local maximum at x_1 if the increment of the function is negative for the sufficiently small increments of the argument.

Here the small increment of the argument means that $x_1 + \Delta x$ has to be in the neighborhood of x_1 mentioned in definition 9.1.

If we denote $x = x_2 + \Delta x$, we can rewrite the condition $f(x) > f(x_2)$ as $f(x_2 + \Delta x) - f(x_2) > 0$ or $\Delta y > 0$.

Conclusion 9.2. The function has local minimum at x_2 if the increment of the function is positive for the sufficiently small increments of the argument.

All local maximums and minimums of a function are called local extrema.

Theorem 9.3 (the necessary condition for existence of local extremum). If the function $f(x)$ has a local extremum at x_0, then $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

Proof. Let us suppose that the function $f(x)$ has a local maximum at x_0. Then by conclusion 9.1 at this point $\Delta y < 0$. If $\Delta x > 0$, then $\dfrac{\Delta y}{\Delta x} < 0$ and by the limit theorem

$$\lim_{\Delta x \to 0+} \frac{\Delta y}{\Delta x} \leq 0. \tag{3.15}$$

If $\Delta x < 0$, then $\dfrac{\Delta y}{\Delta x} > 0$ and by the limit theorem

$$\lim_{\Delta x \to 0-} \frac{\Delta y}{\Delta x} \geq 0. \tag{3.16}$$

If the limit exists, then the one-sided limits (3.15) and (3.16) are equal. This is possible only if they both equal to zero. But then also

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0$$

If the one-sided limits are (3.15) and (3.16) are different, then there does not exist $f'(x_0)$.

Definition 9.3. The point x_0 such that $f'(x_0) = 0$ is called the *stationary point* of the function $f(x)$.

Definition 9.4. We say that x_0 is a critical point of the function $f(x)$ if $f(x_0)$ exists and if either x_0 is a stationary point or $f'(x_0)$ does not exists.

Using the last definition, we can re-phrase the necessary condition for existence of local extremum.

If the function $f(x)$ has a local extremum at x_0, then x_0 is the critical point of the function $f(x)$. That means the only point at which the function has a local extremum is the critical point.

This condition if necessary for existence of a local extremum but not sufficient. The function $y = x^3$ has the derivative of $y' = 3x^2$, which equals to zero if $x = 0$, i.e. $x = 0$ is the critical point of the function $y = x^3$, but this function hasn't local extremum at the point $x = 0$.

Theorem 9.2. Assume that x_0 is a critical point of the function $f(x)$ and the function is differentiable in some neighborhood of x_0 $(x_0 - \varepsilon; x_0 + \varepsilon)$. Then:

1) If in the left-hand neighborhood of x_0 $(x_0 - \varepsilon; x_0)$ $f'(x) > 0$ and in the right-hand neighborhood $(x_0; x_0 + \varepsilon)$ $f'(x) < 0$, then the function $f(x)$ has at x_0 a local maximum.

2) If in the left-hand neighborhood of x_0 $(x_0 - \varepsilon; x_0)$ $f'(x) < 0$ and in the right-hand neighborhood $(x_0; x_0 + \varepsilon)$ $f'(x) > 0$, then the function $f(x)$ has at x_0 a local minimum.

3) If in the left-hand neighborhood of x_0 $(x_0 - \varepsilon; x_0)$ $f'(x) > 0$ and in the right-hand neighborhood $(x_0; x_0 + \varepsilon)$ on $f'(x) > 0$, then the function $f(x)$ is increasing at x_0.

4) If in the left-hand neighborhood of x_0 $(x_0 - \varepsilon; x_0)$ $f'(x) < 0$ and in the right-hand neighborhood $(x_0; x_0 + \varepsilon)$ $f'(x) < 0$, then the function $f(x)$ is decreasing at x_0.

Let us prove the 1st and the 3rd assertion of theorem 9.2.

Proof of the 1st assertion. Fix in the right-hand neighborhood of x_0 one point $x \in (x_0; x_0 + \varepsilon)$. In the interval $[x_0; x]$ there hold all assumptions of Lagrange theorem, i.e. the function $f(x)$ is continuous in closed interval $[x_0; x]$ and differentiable in open interval $(x_0; x)$. By Lagrange theorem there exists $c \in (x_0; x)$ such that

$$f(x) - f(x_0) = f'(c)(x - x_0)$$

As assumed $f'(c) < 0$ and due to the choice of x there holds $x - x_0 > 0$. The product $f'(c)(x - x_0) < 0$, thus, $f(x) < f(x_0)$.

Fix in the left-hand neighborhood of x_0 one point $x \in (x_0 - \varepsilon; x_0)$. In the interval $[x; x_0]$ there hold again all assumptions of Lagrange theorem. Therefore, there exists $c \in (x; x_0)$ such that

$$f(x_0) - f(x) = f'(c)(x_0 - x)$$

By assumption $f'(c) > 0$, according to the choice of x there holds $x_0 - x > 0$. The product $f'(c)(x - x_0) > 0$. Hence, $f(x_0) > f(x)$.

For each fixed $x \in (x_0 - \varepsilon; x_0 + \varepsilon)$ there holds $f(x) < f(x_0)$, that means the function $f(x)$ has the local maximum at x_0.

Proof of the 3rd assertion. Fix in the right-hand neighborhood of x_0 one point $x \in (x_0; x_0 + \varepsilon)$. In thew interval $[x_0; x]$ there hold all the assumptions of Lagrange theorem. It follows, that there exists $c \in (x_0; x)$ such that

$$f(x) - f(x_0) = f'(c)(x - x_0)$$

By assumption $f'(c) > 0$ and the choice of x implies $x - x_0 > 0$. The product $f'(c)(x - x_0) > 0$, hence, $f(x) > f(x_0)$.

Fix in the left-hand neighborhood of x_0 one point $x \in (x_0 - \varepsilon; x_0)$. In the interval $[x; x_0]$ all the assumptions of Lagrange theorem are satisfied. By Lagrange theorem there exists $c \in (x; x_0)$ such that

$$f(x_0) - f(x) = f'(c)(x_0 - x)$$

By assumption $f'(c) > 0$ and the choice of x implies $x_0 - x > 0$. The product $f'(c)(x - x_0) > 0$, that means $f(x_0) > f(x)$.

Therefore, for each $x > x_0$ there holds $f(x) > f(x_0)$ and for each $x < x_0$ there holds $f(x) < f(x_0)$, i.e. the function is increasing in some neighborhood of x_0. The third assertion has been proved.

Let x_0 be a critical point of the function $f(x)$ and draw the assertions of the theorem 9.2 in the following table together.

$x < x_0$	$x > x_0$	assertion
$f'(x) > 0$	$f'(x) < 0$	the function $f(x)$ has at x_0 the local maximum
$f'(x) < 0$	$f'(x) > 0$	the function $f(x)$ has at x_0 the local minimum
$f'(x) > 0$	$f'(x) > 0$	the function is increasing at x_0
$f'(x) < 0$	$f'(x) < 0$	the function is decreasing at x_0

Example 1. Find the local extrema of the function $y = (x-1)\sqrt[3]{x^2}$.

The domain of this function is the set of all real numbers $X = (-\infty; \infty)$. Find the derivative

$$y' = \sqrt[3]{x^2} + (x-1) \cdot \frac{2}{3} x^{-\frac{1}{3}} = \frac{3x + 2(x-1)}{3 \cdot \sqrt[3]{x}} = \frac{5x - 2}{3 \cdot \sqrt[3]{x}}$$

We have that $y' = 0$ if $5x - 2 = 0$, that means $x = \frac{2}{5}$ and y' does not exist if $x = 0$. The critical points are $x_1 = 0$ and $x_2 = \frac{2}{5}$.

If $x < 0$, then $5x - 2 < 0$ and $\sqrt[3]{x} < 0$, hence, $y' > 0$.

If $0 < x < \frac{2}{5}$, then $5x - 2 < 0$ and $\sqrt[3]{x} > 0$, hence, $y' < 0$.

If $x > \frac{2}{5}$, then $5x - 2 > 0$ and $\sqrt[3]{x} > 0$ and $y' > 0$.

On the left side of $x_1 = 0$ the derivative $y' > 0$ and on the right side $y' < 0$. According to the theorem 9.2 the given function has at this point the local maximum.

On the left side of $x_2 = \frac{2}{5}$ the derivative $y' < 0$ and on the right side $y' > 0$. Thus, given function has at the point $x_2 = \frac{2}{5}$ the local minimum.

3.10 Second derivative test

In mathematical analysis, the second derivative test is a criterion for determining whether a real function of one variable has at the critical point a local maximum or a local minimum using the value of the second derivative at this point.

Let x_0 be a stationary point of the function $f(x)$, i.e. $f'(x_0) = 0$.

Theorem 10.1. Let $f''(x)$ be defined and continuous in some neighborhood of the stationary point x_0. If $f''(x_0) < 0$ then the function $f(x)$ has a local maximum at x_0. If $f''(x_0) > 0$ then the function $f(x)$ has a local minimum at x_0.

If $f''(x_0) = 0$ then this theorem is inconclusive.

Proof. Because of continuity of $f''(x)$ the second derivative $f''(x) < 0$ in some neighborhood of x_0, that means in this neighborhood $f'(x)$ is decreasing. By the assumption $f'(x_0) = 0$, therefore, if $x < x_0$, then $f'(x) > 0$ and if $x > x_0$ then $f'(x) < 0$. Due to the theorem 9.2 the function $f(x)$ has a local maximum at x_0.

The second assertion of this theorem can be proved analogously.

Example 10.1. Find the local extrema of the function $y = 2\sin x + \cos 2x$ using the second derivative test.

First we find the derivative $y' = 2\cos x - 2\sin 2x = 2\cos x(1 - 2\sin x)$. The stationary points we find solving the equation

$$2\cos x(1 - 2\sin x) = 0$$

If $2\cos x = 0$ then $x = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$. If $1 - 2\sin x = 0$ then $\sin x = \frac{1}{2}$ then $x = (-1)^n \frac{\pi}{6} + n\pi$, $n \in \mathbb{Z}$. All together we have two sets of stationary points.

Next we find the second derivative $y'' = -2\sin x - 4\cos 2x$. Evaluating the value of second derivative at the points $x = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$, we obtain $y'' = -2\sin\left(\frac{\pi}{2} + n\pi\right) - 4\cos 2\left(\frac{\pi}{2} + n\pi\right)$.

The equalities $\sin\left(\frac{\pi}{2} + n\pi\right) = (-1)^n$ and $\cos(\pi + 2n\pi) = \cos\pi = -1$

yield that $y''\left(\frac{\pi}{2}+n\pi\right) = -2(-1)^n + 4 > 0$ for each $n \in \mathbb{Z}$. Consequently at the points $x = \frac{\pi}{2} + n\pi$ $n \in \mathbb{Z}$ the given function has the local minima.

At the points $x = (-1)^n\frac{\pi}{6} + n\pi$, $n \in \mathbb{Z}$ we obtain that

$$\begin{aligned} y'' &= -2\sin\left((-1)^n\frac{\pi}{6}+n\pi\right) - 4\cos 2\left((-1)^n\frac{\pi}{6}+n\pi\right) \\ &= -2\cdot\frac{1}{2} - 4\cos\left((-1)^n\frac{\pi}{3}+2n\pi\right) = -1-4\cos\left((-1)^n\frac{\pi}{3}\right) = -1-4\cdot\frac{1}{2} = -3. \end{aligned}$$

Therefore, at the points $x = (-1)^n\frac{\pi}{6} + n\pi$, $n \in \mathbb{Z}$ the given function has the local maxima.

3.11 Greatest and least value of function in closed interval

Sometimes the greatest and the least value of the function are also referred as the global maximum and global minimum. Together as global extrema.

Let the function $y = f(x)$ be continuous in the closed interval $[a; b]$. The finding of the greatest and the least value of the function in the given interval is grounded on two facts.

1. The continuous on a closed interval function has the least and the greatest value on that interval.

2. The continuous on a closed interval function acquires the least and the greatest value either at the critical point or at the endpoint of that interval.

From these two assertions we conclude the instruction for finding the least and the greatest values of the function $y = f(x)$ in the closed interval $[a; b]$.

1. Find the critical points x_1, x_2, ... of the function $y = f(x)$ in the closed interval $[a; b]$ and the values of the function at these points $f(x_1)$, $f(x_2)$, ...

2. Find the values of the function at the endpoints of this closed interval $f(a)$ and $f(b)$.

3. Choose the greatest y_{max} and the least y_{min} of the values found.

Example 11.1. Find the greatest and the least value (the global extrema) of the function $y = x^4 - 4x^3 - 20x^2$ in the closed interval $[-3; 1]$.

The derivative is $y' = 4x^3 - 12x^2 - 40x$ and the critical points we obtain solving the equation $4x^3 - 12x^2 - 40x = 0$. Dividing by 4 and factoring gives the equation $x(x + 2)(x - 5) = 0$, whose solutions (i.e. the critical points) are $x_1 = -2$, $x_2 = 0$ and $x_3 = 5$. The values of the function at the first two critical points are $f(-2) = -32$ and $f(0) = 0$. We do not evaluate the function at the third critical point because this is located outside the given interval and therefore is outside our interests. The values of the function at the endpoints are $f(-3) = 9$ and $f(1) = -23$. Now we choose from the values obtained the greatest and the least

$$y_{max} = y(-3) = 9$$

and

$$y_{min} = y(-2) = -32$$

3.12 Convexity and concavity of graph of function. Inflection points

Definition 12.1. The graph of the function is called *convex* in the interval $(a; b)$ if any tangent line drawn to the graph in this interval is not below the graph. The interval $(a; b)$ is called the *interval of convexity* and denoted by $\hat{X}$.

Definition 12.2. The graph of the function is called *concave* in the interval $(a; b)$ if any tangent line drawn to the graph in this interval is not above the graph. The interval $(a; b)$ is called the *interval of concavity* and denoted by $\check{X}$.

Definition 12.3. The point on the graph of the function separating the intervals of convexity and concavity is called *inflection point.*

Conclusion. At the inflection point the tangent line intersects the graph of the function because in one side of this point the tangent line is not below the graph and in other side of this point not above the graph.

Theorem 12.1. Suppose the function $y = f(x)$ has in the interval $(a; b)$ continuous first and second order derivatives. If $f''(x) < 0$ in the interval $(a; b)$, then the graph of this function is convex in this interval.

Proof. Let us fix $x_0 \in (a; b)$ and sketch the tangent line to the graph of the function at the point $P_0(x_0; f(x_0))$. Let us have one more arbitrarily chosen $x \in (a; b)$ such that $x \neq x_0$. Denote the ordinate of the corresponding point on the tangent line $\bar{y}$. The equation of the line tangent to the graph of $y = f(x)$ at x_0 is $\bar{y} = f(x_0) + f'(x_0)(x - x_0)$.

Then

$$\bar{y} - f(x) = f(x_0) + f'(x_0)(x - x_0) - f(x) = -[f(x) - f(x_0)] + f'(x_0)(x - x_0)$$

On the closed interval $[x_0; x]$ there hold all the assumptions of Lagrange theorem, therefore, there exists $\bar{x} \in (x_0; x)$ such that $f(x)-f(x_0) = f'(\bar{x})(x-x_0)$. Thus,

$$\bar{y} - f(x) = -f'(\bar{x})(x - x_0) + f'(x_0)(x - x_0) = -(x - x_0)(f'(\bar{x}) - f'(x_0))$$

The function $f'(x)$ also satisfies all the assumptions of Lagrange theorem on the closed interval $[x_0; \bar{x}]$. By Lagrange theorem there exists $\xi \in (x_0; \bar{x})$ such that $f'(\bar{x}) - f'(x_0) = f''(\xi)(\bar{x} - x_0)$. Consequently

$$\bar{y} - f(x) = -f''(\xi)(x - x_0)(\bar{x} - x_0)$$

If $x > x_0$, then $x - x_0 > 0$ and as $x_0 < \bar{x} < x$ then $\bar{x} - x_0 > 0$ and the product $(x - x_0)(\bar{x} - x_0) > 0$. As assumed $f''(x) < 0$, therefore, $\bar{y} - f(x) > 0$ or $\bar{y} > f(x)$

If $x < x_0$ then $x - x_0 < 0$ and as $x < \bar{x} < x_0$ then $\bar{x} - x_0 < 0$. The product $(x - x_0)(\bar{x} - x_0) > 0$, hence, $\bar{y} > f(x)$.

Thus, the value of the ordinate of the point on tangent line with any abscissa $x \neq x_0$, $x \in (a; b)$, is greater than the value of the ordinate of the corresponding point on the graph of the function, that means the point on the tangent line is above the corresponding point of the graph of the function. According to the definition of the convexity the graph is convex.

In the similar way can be proved.

Theorem 12.2. Suppose the function $y = f(x)$ has in the interval $(a; b)$ continuous first and second order derivatives. If $f''(x) > 0$ in the interval $(a; b)$ then the graph of this function is concave in this interval.

Theorem 12.3. Suppose the function $f(x)$ has at x_0 the derivative $f'(x_0)$ or has a vertical tangent line at x_0, $f''(x_0) = 0$ or $f''(x_0)$ does not exist and $f''(x)$ changes it's sing at x_0 then the point $(x_0; f(x_0)$ is an inflection point of the graph of the given function.

Example 12.1. Find the intervals of convexity and concavity and the inflection points of the graph of the function $y = e^{-x^2}$.

First find the first derivative $y' = -2xe^{-x^2}$ and then the second derivative $y'' = -2e^{-x^2} + 4x^2e^{-x^2} = 2e^{-x^2}(2x^2 - 1)$.

Because of $2e^{-x^2} > 0$, the equation

$$2e^{-x^2}(2x^2 - 1) = 0$$

is equivalent to the equation $2x^2-1=0$, whose solutions are $x_1=-\dfrac{1}{\sqrt{2}}$ and $x_2=\dfrac{1}{\sqrt{2}}$.

To find the interval of convexity we solve the inequality $2x^2-1<0$ and obtain $-\dfrac{1}{\sqrt{2}}<x<\dfrac{1}{\sqrt{2}}$. The intervals of concavity we find as the solutions of the inequality $2x^2-1>0$, which are $x<-\dfrac{1}{\sqrt{2}}$ or $x>\dfrac{1}{\sqrt{2}}$. The second derivative changes the sing at both values of x found. The value of the function at $x=-\dfrac{1}{\sqrt{2}}$ is $y=e^{-\frac{1}{2}}$ and the value of the function at $x=\dfrac{1}{\sqrt{2}}$ is also $y=e^{-\frac{1}{2}}$.

Thus, the interval of convexity of the given function is $\hat{X}=\left(-\dfrac{1}{\sqrt{2}};\dfrac{1}{\sqrt{2}}\right)$, the intervals of concavity are $\check{X}=\left(-\infty;-\dfrac{1}{\sqrt{2}}\right)$ and $\check{X}\left(\dfrac{1}{\sqrt{2}};\infty\right)$ and the inflection points are $K_1\left(-\dfrac{1}{\sqrt{2}};\dfrac{1}{\sqrt{e}}\right)$ and $K_2\left(\dfrac{1}{\sqrt{2}};\dfrac{1}{\sqrt{e}}\right)$

3.13 Asymptotes of graph of function

Let O be the origin of the coordinate plane and $M(x;y)$ some point on the graph of the function $y=f(x)$.

Definition 13.1. It is said that the point of the graph of the function tends to infinity if the length of the position vector $\overrightarrow{OM}$ of the point M increases unboundedly, i.e. $|\overrightarrow{OM}|=\sqrt{x^2+y^2}\to\infty$.

Definition 13.2. The line is called the asymptote of the graph of the function if the distance of a point of the graph form the line is an infinitesimal as the point of the graph tends to infinity.

There are two kinds of asymptotes: the vertical asymptotes and slant (oblique) asymptotes.

The equation of a vertical line is $x=a$. This line is the vertical asymptote of the graph of the function if the distance of the point $M(x;y)$ from this line is an infinitesimal if M tends to infinity, i.e. $\lim\limits_{|\overrightarrow{OM}|\to\infty}|x-a|=0$.

The equalities $\lim\limits_{|\overrightarrow{OM}|\to\infty}|\overrightarrow{OM}|=\sqrt{x^2+y^2}=\lim\limits_{|\overrightarrow{OM}|\to\infty}\sqrt{a^2+y^2}=\infty$ are possible only if $\lim|y|=\infty$.

Therefore, the line $x=a$ can be the vertical asymptote of the graph of the function $y=f(x)$ if

$$\lim_{x\to a-}f(x)=\pm\infty$$

or

$$\lim_{x \to a+} f(x) = \pm\infty$$

In other words: the line $x = a$ is the vertical asymptote of the graph of the function $y = f(x)$ if this function has at $x = a$ an infinite discontinuity.

If the line, which is the asymptote of the graph of the function, is not vertical then its angle of elevation $\varphi \neq \frac{\pi}{2}$, the slope of this line $k = \tan\varphi$ is finite and the equation of the slant asymptote is $y = kx + b$. Let us deduce the formulas for finding the slope and the intercept of the slant asymptote from the function $y = f(x)$.

If the distance of the point $M(x; y)$ of the graph of $y = f(x)$ from the line $y = kx + b$ is an infinitesimal as M tends to infinity then $|x| \to \infty$ (if not, we have the vertical asymptote).

Let M be the point on the graph of the function $y = f(x)$ (Figure 3.3). The distance of that point from the line $y = kx + b$ is the length of MP. By assumption the line $y = kx + b$ is the asymptote of the graph of the function $y = f(x)$. Thus by the definition 13.2

$$\lim_{|x| \to \infty} MP = 0. \tag{3.17}$$

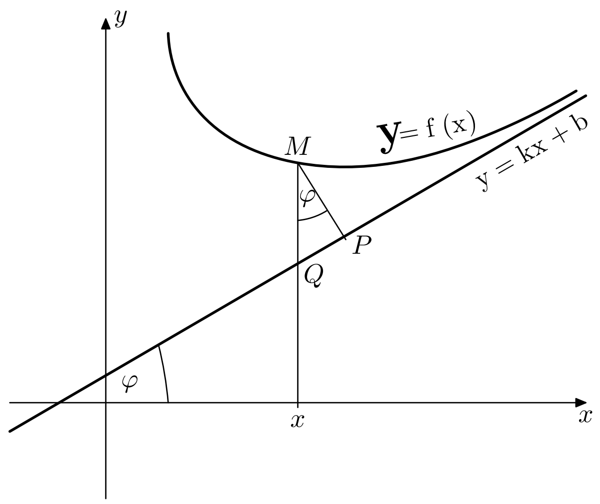

Figure 3.4: the slant asymptote of the graph of the function

Obviously $\angle PMQ = \varphi$ (the correspondent sides are perpendicular). As $\varphi \neq \frac{\pi}{2}$ then $\cos\varphi \neq 0$ and due to equality $MP = MQ \cdot \cos\varphi$ the condition (3.17) yields $\lim\limits_{|x| \to \infty} MQ = 0$.

But $MQ = |f(x) - (kx + b)|$. Hence,

$$\lim_{|x|\to\infty} (f(x) - kx - b) = 0. \tag{3.18}$$

or

$$\lim_{|x|\to\infty} \left[x \left(\frac{f(x)}{x} - k - \frac{b}{x} \right) \right] = 0$$

As $|x| \to \infty$ then the last condition is satisfied only if

$$\lim_{|x|\to\infty} \left(\frac{f(x)}{x} - k - \frac{b}{x} \right) = 0$$

Because of equality $\lim_{|x|\to\infty} \frac{b}{x} = 0$ the last condition implies

$$\lim_{|x|\to\infty} \left(\frac{f(x)}{x} - k \right) = 0$$

which yields

$$k = \lim_{|x|\to\infty} \frac{f(x)}{x} \tag{3.19}$$

The equality (3.18) implies

$$b = \lim_{|x|\to\infty} (f(x) - kx). \tag{3.20}$$

Now we formulate the result as a theorem.

Theorem 13.1. The line $y = kx + b$ is the slant asymptote of the graph of the function $y = f(x)$ if and only if the slope k and the intercept b are evaluated by formulas (3.19) and (3.20) respectively provided the limits in these formulas exist.

Example 13.1. Find the asymptotes of the graph of the function $y = \frac{x^2}{x - 1}$.

The function is discontinuous at $x = 1$. Evaluating the one-sided limits

$$\lim_{x\to 1-} \frac{x^2}{x - 1} = -\infty$$

and

$$\lim_{x\to 1+} \frac{x^2}{x - 1} = \infty$$

we obtain that the line $x = 1$ is the vertical asymptote of the graph of the given function.

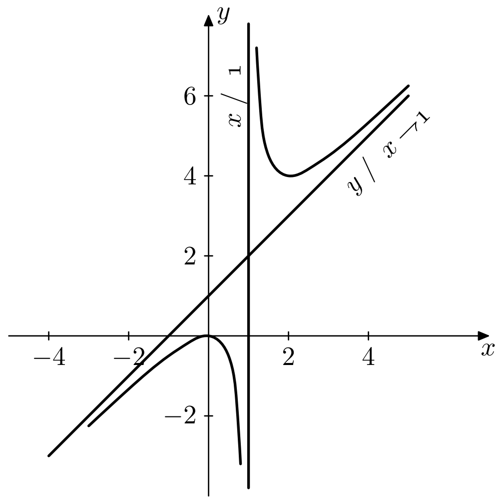

Figure 3.5: the graph and the asymptotes of the function $y = \dfrac{x^2}{x-1}$

The slope of the slant asymptote we evaluate by (3.19)

$$k = \lim_{x\to\infty} \frac{x^2}{x-1} : x = \lim_{x\to\infty} \frac{x^2}{x^2 - x} = 1$$

and the intercept by (3.20)

$$\lim_{x\to\infty} \left(\frac{x^2}{x-1} - x \right) = \lim_{x\to\infty} \frac{x^2 - x^2 + x}{x-1} = \lim_{x\to\infty} \frac{x}{x-1} = 1$$

Thus, the slant asymptote of the graph of this function is $y = x + 1$.

The graph of the function $y = \dfrac{x^2}{x-1}$ and the asymptotes are drawn in Figure 3.4.