5 Indefinite integral

The most of the mathematical operations have inverse operations: the inverse operation of addition is subtraction, the inverse operation of multiplication is division, the inverse operation of exponentation is rooting. The inverse operation of differentiation is called integration. For example, describing a process at the given moment knowing the speed of this process at that moment.

5.1 Definition and properties of indefinite integral

The function $F(x)$ is called an *antiderivative* of $f(x)$ if $F'(x) = f(x)$. For example, the antiderivative of x is $\frac{x^2}{2}$ because $(\frac{x^2}{2})' = x$, the antiderivative of $\cos x$ is $\sin x$ because $(\sin x)' = \cos x$ etc. An antiderivative is not uniquely determined because after $\sin x$ the antiderivatives of $\cos x$ are also $\sin x + 2$, $\sin x - \pi$ and any function $\sin x + C$, where C is an arbitrary constant.

More generally, if $F(x)$ is an antiderivative of $f(x)$ then the antiderivative of $f(x)$ is also every function $F(x) + C$, where C is whatever constant. The question is: has the function $f(x)$ some other antiderivatives that are different from $F(x) + C$. The next two corollaries give the answer to this question.

Corollary 1.1. If $F'(x) = 0$ in some interval $(a; b)$ then $F(x)$ is constant in that interval.

Proof. Let us fix a point $x \in (a; b)$ and choose whatever Δx so that $x + \Delta x \in (a; b)$ According to Lagrange theorem there exists $\xi \in (x; x + \Delta x)$ such that

$$F(x + \Delta x) - F(x) = F'(\xi)\Delta x$$

We have assumed that the derivative of $F(x)$ equals to zero in the interval $(a; b)$, therefore, $F'(\xi) = 0$ that means $F(x+\Delta x)-F(x) = 0$ or $F(x+\Delta x) = F(x)$ for whatever Δx. Consequently, the value at any $x+\Delta x \in (a; b)$ equals to the value at a fixed point $x \in (a; b)$ which means that this is a constant function.

Corollary 1.2. Is $F(x)$ and $G(x)$ are two antiderivatives of the function $f(x)$ then they differ at most by a constant.

Proof. As assumed $F'(x) = f(x)$ and $G'(x) = f(x)$. Thus,

$$[G(x) - F(x)]' = G'(x) - F'(x) = f(x) - f(x) = 0$$

and by corollary 1.1 $G(x) - F(x) = C$, where C is an arbitrary constant, or $G(x) = F(x) + C$. This result means that any antiderivative, which is different from $F(x)$ cam be expressed as $F(x) + C$.

We can sum up in the following way: if the function $F(x)$ is an antiderivative of $f(x)$ then each function $F(x) + C$ is also an antiderivative and there exist no antiderivatives in different form. This gives us the possibility to define.

Definition 1.3. If the function $F(x)$ is an antiderivative of $f(x)$ then the expression $F(x)+C$, where C is an arbitrary constant, is called the *indefinite integral* of $f(x)$ with respect to x and denoted $\int f(x)dx$.

By this definition

$$\int f(x)dx = F(x) + C$$

The function $f(x)$ is called the *integrand*, $\displaystyle\int$ the *integral sign*, x is called the *variable of integration* and C the *constant of integration*.

Using the examples considered, we can write now that

$$\int \cos x dx = \sin x + C$$

and

$$\int x dx = \frac{x^2}{2} + C$$

We make some conclusions from the definition 1.3.

Conclusion 1.4. $\left(\int f(x)dx\right)' = (F(x) + C)' = f(x)$, that means the derivative of the indefinite integral equals to the integrand.

Conclusion 1.5. The differential of the indefinite integral

$$d\left(\int f(x)dx\right) = \left(\int f(x)dx\right)' dx = f(x)dx$$

is the expression under integral sign.

Conclusion 1.6. $\int dF(x) = F(x) + C$, i.e. the indefinite integral of the differential of a function equals to the sum of that function and an arbitrary constant. Indeed, as $F'(x) = f(x)$ then

$$\int dF(x) = \int F'(x)dx = \int f(x)dx = F(x) + C$$

5.2 Table of basic integrals

The integral of the power function

2.1. $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad \alpha \in \mathbb{R},\ \alpha \neq -1$

and three special cases of this formula

$\int dx = x + C,$

$\int \frac{dx}{x^2} = -\frac{1}{x} + C,$

$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C.$

The first special case is included in the general formula if the exponent $\alpha = 0$, the second if $\alpha = -2$ and the third if $\alpha = -\frac{1}{2}$

If in the indefinite integral of power function $\alpha = -1$ then

2.2. $\int \frac{dx}{x} = \ln|x| + C$

The indefinite integrals of trigonometric functions

2.3. $\int \cos x dx = \sin x + C$

2.4. $\int \sin x dx = -\cos x + C$

2.5. $\int \frac{dx}{\cos^2 x} = \tan x + C$

2.6. $\int \frac{dx}{\sin^2 x} = -\cot x + C$

The indefinite integral of the exponential function

2.7. $\int a^x dx = \frac{a^x}{\ln a} + C, \quad a > 0,\ a \neq 1$

and if the base $a = e$ then $\int e^x dx = e^x + C$

The indefinite integrals concerning the inverse trigonometric functions

2.8. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

2.9. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C$

2.10. $\int \frac{dx}{1+x^2} = \arctan x + C$

2.11. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$

Two indefinite integrals containing natural logarithms

2.12. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C.$

2.13. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C$

The indefinite integrals of hyperbolic functions

2.14. $\displaystyle\int \sinh x dx = \cosh x + C$

2.15. $\displaystyle\int \cosh x dx = \sinh x + C$

2.17. $\displaystyle\int \frac{dx}{\sinh^2 x} = -\coth x + C$

2.17. $\displaystyle\int \frac{dx}{\cosh^2 x} = \tanh x + C$

All of these formulas can be directly proved by differentiating the right side of the equalities (for the reader it is useful to check the formulas 2.12 and 2.13).

5.3 Properties of indefinite integral

Next we shall prove three properties of the indefinite integrals and use them to integrate some functions.

Property 3.1. $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$, i.e. the indefinite integral of the sum equals to the sum of the indefinite integrals.

Proof. Two indefinite integrals are equal if the set of antiderivatives is the same, i.e the derivatives are equal. By the conclusion 1.4 the derivative of the left side

$$\left(\int [f(x) + g(x)]dx\right)' = f(x) + g(x)$$

To find the derivative of the right side we use the sum rule of the derivative and the conclusion 1.4 again

$$\left(\int f(x)dx + \int g(x)dx\right)' = \left(\int f(x)dx\right)' + \left(\int g(x)dx\right)' = f(x) + g(x)$$

Property 3.2. If a is a constant then $\int af(x)dx = a\int f(x)dx$, i.e. the constant coefficient can be carried outside the sign of integral.

This property can be proved in similar way as the property 3.1.

Property 3.3. $\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$, i.e. the indefinite integral of the difference of two functions is equal to the difference of indefinite integrals of these functions.

This turns out from the two previous properties

$$\begin{aligned}\int [f(x) - g(x)]dx = \int [f(x) + (-1)g(x)]dx &= \int f(x)dx + \int (-1)g(x)dx \\ &= \int f(x)dx - \int g(x)dx\end{aligned}$$

Let us have some examples of indefinite integrals that can be found, using three properties and the table of basic integrals.

Example 3.4. Find $\int (x^2 + 2\sin x)\, dx$.

Using the properties 3.1 and 3.2 and the basic integrals 2.1 and 2.4, we have

$$\int x^2 dx + \int 2\sin x dx = \int x^2 dx + 2\int \sin x dx = \frac{x^3}{3} - 2\cos x + C$$

Example 3.5. Find

$$\int \frac{(x-1)^2}{x(1+x^2)} dx$$

Here we first remove the parenthesis in the numerator, then divide by terms, use the properties 3.3 and 3.2 and basic integrals 2.2 and 2.10:

$$\begin{aligned}\int \frac{(x-1)^2}{x(1+x^2)} dx &= \int \frac{x^2+1-2x}{x(1+x^2)} dx = \int \left(\frac{x^2+1}{x(1+x^2)} - \frac{2x}{x(1+x^2)} \right) dx \\ &= \int \left(\frac{1}{x} - \frac{2}{1+x^2} \right) dx = \int \frac{dx}{x} - 2\int \frac{dx}{1+x^2} \\ &= \ln|x| - 2\arctan x + C\end{aligned}$$

Example 3.6. Find

$$\int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx$$

First we use the formula of the cosine of the double angle, then divide by terms, next the property 3.3 and finally the basic integrals 2.5 and 2.6. We obtain that

$$\begin{aligned}\int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx &= \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{\cos^2 x}{\sin^2 x \cos^2 x} - \frac{\sin^2 x}{\sin^2 x \cos^2 x} \right) dx \\ &= \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} \right) dx = \int \frac{dx}{\sin^2 x} - \int \frac{dx}{\cos^2 x} = -\cot x - \tan x + C\end{aligned}$$

Remark. So far we have used in the role of the variable of integration only x. Naturally, we can use in this role any notation. Instead of $\int f(x)dx$ we can integrate $\int f(y)dy$, $\int f(t)dt = \dots$.

We can integrate a lot of functions, using three properties of the indefinite integral, the table of basic integrals an elementary transformations of the given function. But we can significantly enlarge the amount of functions to be integrated using some technique of integration such as change of variable, integration by parts etc.

5.4 Integration by changing variable

Consider the indefinite integral $\displaystyle\int f(x)dx$ and one-valued differentiable function $x = g(t)$, which has one- valued inverse function $t = g^{-1}(x)$

Theorem 4.1. If $x = g(t)$ is strictly increasing (strictly decreasing) differentiable function then

$$\int f(x)dx = \int f[g(t)]g'(t)dt \tag{4.1}$$

Proof. We use again the fact that the indefinite integrals are equal if the derivatives of these are equal. Let us differentiate with respect to x both sides of the equality (4.1) and become convinced that the result is the same.

By conclusion 1.4 the derivative of the left side of (4.1) is $f(x)$. The antiderivative of the right side is a function of the variable t. To differentiate this with respect to x we have to use the chain rule: the derivative of the antiderivative with respect to t times the derivative of t with respect to x:

$$\frac{d}{dx}\left(\int f\left[g(t)\right]g'(t)dt\right) = \frac{d}{dt}\left(\int f\left[g(t)\right]g'(t)dt\right)\cdot\frac{dt}{dx}.$$

By conclusion 1.4

$$\frac{d}{dt}\left(\int f\left[g(t)\right]g'(t)dt\right) = f\left[g(t)\right]g'(t)$$

As assumed $g'(t) \neq 0$, thus the derivative of the inverse function $t = g^{-1}(x)$ is the reciprocal of the derivative of the given function

$$\frac{dt}{dx} = \frac{d}{dx}(g^{-1}(x)) = \frac{1}{g'(t)}.$$

All together

$$\frac{d}{dx}\left(\int f\left[g(t)\right]g'(t)dt\right) = f\left[g(t)\right]g'(t)\cdot\frac{1}{g'(t)} = f[g(t)] = f(x)$$

Indeed, the derivatives of both sides of the equality (4.1) are equal to $f(x)$, which proves the assertion of this theorem.

The goal of the change of the variable is to obtain the indefinite integral which is in the table of integrals or can be found, using some transformations or some other technique of integration. For example let us assume that the table of basic integrals contains $\int f(x)dx = F(x) + C$ and we have to find the integral

$$\int f(g(x))g'(x)dx$$

Changing the variable $t = g(x)$, the differential $dt = g'(x)$ and the integral transforms

$$\int f(g(x))g'(x)dx = \int f(t)dt = F(t) + C = F(g(x)) + C$$

Example 4.2. Find

$$\int \frac{xdx}{\sqrt{x^2+1}}.$$

To find this integral, we use the change of variable $t = x^2 + 1$ (also we could use the same change of variable $x = \sqrt{t-1}$). Then the differential $dt = 2xdx$ or $xdx = \frac{dt}{2}$ and

$$\int \frac{x}{\sqrt{x^2+1}}dx = \int \frac{1}{\sqrt{t}}\frac{dt}{2} = \sqrt{t} + C = \sqrt{x^2+1} + C.$$

Two conclusions from the theorem 4.1.

Conclusion 4.3.

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C,$$

i.e. the indefinite integral of a fraction, whose numerator is the derivative of the denominator, equals to the natural logarithm of the absolute value of the denominator plus the constant of integration.

Indeed, changing the variable $t = f(x)$ in the integral $\int \frac{f'(x)}{f(x)}dx$, we have $dt = f'(x)dx$ and

$$\int \frac{f'(x)}{f(x)}dx = \int \frac{dt}{t} = \ln|t| + C = \ln|f(x)| + C$$

Example 4.4. Find

$$\int \cot x dx = \int \frac{\cos x}{\sin x}dx = \ln|\sin x| + C.$$

Example 4.5. Find

$$\int \frac{xdx}{x^2+1} = \frac{1}{2}\int \frac{2x}{x^2+1}dx = \frac{1}{2}\ln(x^2+1) + C.$$

Here we need not to use the absolute value of the argument of natural logarithm x^2+1 because this is positive for any real value of x.

Example 4.6. Find

$$\int \coth x dx = \int \frac{\cosh x}{\sinh x} dx = \ln|\sinh x| + C.$$

Conclusion 4.7. If $\int f(x)dx = F(x) + C$, then for any $a \neq 0$

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C,$$

i.e. if the argument x of the integrable function has been substituted with a linear expression $ax+b$ then the argument of the antiderivative is also $ax+b$ and the antiderivative is multiplied by the reciprocal $1/a$ of the coefficient of x.

To verify the assertion of conclusion 4.7 it is sufficient to change the variable $t = ax+b$, which yields $dt = adx$ or $dx = \frac{1}{a}dt$ and then

$$\int f(ax+b)dx = \int f(t)\frac{1}{a}dt = \frac{1}{a}\int f(t)dt = \frac{1}{a}F(t) + C = \frac{1}{a}F(ax+b) + C$$

Example 4.8. Knowing the integral $\int \cos x dx = \sin x + C$ and using conclusion 4.7, we obtain that

$$\int \cos(3x+4)dx = \frac{1}{3}\sin(3x+4) + C$$

Example 4.9. Knowing the integral $\int \frac{dx}{\cos^2 x} = \tan x + C$, we find

$$\int \frac{dx}{\cos^2 \frac{x}{3}} = \frac{1}{\frac{1}{3}}\tan\frac{x}{3} + C = 3\tan\frac{x}{3} + C$$

Example 4.10. Knowing (2.11 of the table of integrals) $\int \frac{dx}{2+x^2} = \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + C$, we find

$$\int \frac{dx}{4x^2+4x+3} = \int \frac{dx}{(2x+1)^2+2} = \frac{1}{2\sqrt{2}}\arctan\frac{2x+1}{\sqrt{2}} + C$$

In some cases there is no need to use the new variable. Let us suppose that $\int f(x)dx$ is in the table of integrals and we have to find

$$\int \varphi'(x)f[\varphi(x)]dx$$

The differential of the function $\varphi(x)$ is $d[\varphi(x)] = \varphi'(x)dx$ and we can rewrite

$$\int \varphi'(x)f[\varphi(x)]dx = \int f[\varphi(x)]d[\varphi(x)]$$

The last integral is the same as $\int f(x)dx$ but instead of the variable x there is the variable $\varphi(x)$. If we use the table of integrals, we substitute variable x by $\varphi(x)$.

Example 4.11. Using the differential $d(x^2+2) = 2xdx$ or $xdx = \frac{1}{2}d(x^2+2)$ and 2.1 in the table of basic integrals, we find

$$\int x\sqrt{x^2+2}dx = \frac{1}{2}\int \sqrt{x^2+2}d(x^2+2) = \frac{1}{2}\frac{(x^2+2)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{(x^2+2)\sqrt{x^2+2}}{3} + C$$

Example 4.12. Using the differential $d(x^3) = 3x^2dx$ and 2.4 in the table, we find

$$\int x^2 \sin x^3 dx = \frac{1}{3}\int 3x^2 \sin x^3 dx = \frac{1}{3}\int \sin x^3 d(x^3) = -\frac{1}{3}\cos x^3 + C$$

5.5 Integration by parts

The differential of the product of two differentiable functions $u = u(x)$ and $v = v(x)$ is

$$d[u(x)v(x)] = [u(x)v(x)]'dx = [u'(x)v(x)+u(x)v'(x)]dx = u(x)v'(x)dx+u'(x)v(x)dx = udv+vdu$$

The property of the indefinite integral 3.1 yields

$$\int d(uv) = \int udv + \int vdu$$

and by conclusion 1.6

$$uv = \int udv + \int vdu$$

The last equality implies the formula of *integration by parts*

$$\int udv = uv - \int vdu \qquad (5.1)$$

Here we have two questions. First, what kind of functions have to be integrated by parts and second, what we choose as $u(x)$ and what we choose as $dv = v'(x)dx$. Integration by parts is not only a purely mechanical process for solving integrals; given a single function to integrate, the typical strategy is to carefully separate it into a product of a function $u(x)$ and the differential $v'(x)dx$ such that the integral produced by the integration by parts formula is easier to evaluate than the original one. It is useful to choose u as a function that simplifies when differentiated, and/or to choose v' as a function that simplifies when integrated.

Integration by parts can be applied to very various classes of functions included the functions that can be integrated using some other technique of integration. More interesting are the functions that can be integrated only by parts. Those are, for example:

1) the products of polynomials and sine function,
2) the products of polynomials and cosine function,
3) the products of polynomials and exponential function.

In any of these three cases we choose the polynomial as u and the product of sine function and dx (cosine or exponential function and dx - respectively) as dv.

Example 5.1. Find $\displaystyle\int (x^2 + 3x) \sin 2x dx$.

Here the integrand is the product of the polynomial and the sine function. Thus, we choose in formula (5.1) $u = x^2 + 3x$ and $dv = \sin 2x dx$. Next we find the differential $du = (2x + 3)dx$ and, using the conclusion 4.7, $v = \int \sin 2x dx = -\dfrac{1}{2} \cos 2x$. Finding the function v the constant of integration is reasonable to take equal to zero because otherwise the terms with that constant reduce anyway. It is useful to check it oneself. Now, by (5.1) we obtain

$$\begin{aligned}\int (x^2 + 3x) \sin 2x dx &= -\frac{1}{2}(x^2 + 3x) \cos 2x - \int (2x + 3) \left(-\frac{1}{2}\right) \cos 2x dx \\ &= -\frac{1}{2}(x^2 + 3x) \cos 2x + \frac{1}{2} \int (2x + 3) \cos 2x dx.\end{aligned}$$

The last integrand is the product of polynomial and cosine function. This has to be integrated by parts again choosing $u = 2x + 3$ and $dv = \cos 2x dx$. Then $du = 2dx$ and $v = \int \cos 2x dx = \dfrac{1}{2} \sin 2x$. By (5.1)

$$\begin{aligned}\int (2x + 3) \cos 2x dx &= \frac{1}{2}(2x + 3) \sin 2x - \int \frac{1}{2} \sin 2x \cdot 2dx = \\ &= \frac{2x + 3}{2} \sin 2x - \int \sin 2x dx = \frac{2x + 3}{2} \sin 2x + \frac{1}{2} \cos 2x + C.\end{aligned}$$

Finally,

$$\begin{aligned}\int (x^2+3x)\sin 2x dx &= -\frac{1}{2}(x^2+3x)\cos 2x + \frac{2x+3}{4}\sin 2x + \frac{1}{4}\cos 2x + C\\ &= \frac{1-6x-2x^2}{4}\cos 2x + \frac{2x+3}{4}\sin 2x + C.\end{aligned}$$

Example 5.2. Find $\displaystyle\int x5^x dx$.

Here the integrand is the product of the polynomial and the exponential function. Choosing $u = x$ and $dv = 5^x dx$, we have $du = dx$ and $v = \displaystyle\int 5^x dx = \frac{5^x}{\ln 5}$. Integration by parts gives

$$\begin{aligned}\int x5^x dx &= x\frac{5^x}{\ln 5} - \int \frac{5^x}{\ln 5}dx = \frac{x5^x}{\ln 5} - \frac{1}{\ln 5}\int 5^x dx\\ &= \frac{x5^x}{\ln 5} - \frac{1}{\ln 5}\cdot\frac{5^x}{\ln 5} + C = \frac{5^x}{\ln 5}\left(x - \frac{1}{\ln 5}\right) + C.\end{aligned}$$

Beside of the products given above there is the huge variety of functions that can be integrated only by parts.

Example 5.3. Find $\displaystyle\int x\log x dx$.

Using the formula (5.1), we choose $u = \log x$ and $dv = xdx$. Hence, $du = \dfrac{dx}{x\ln 10}$ and $v = \int xdx = \dfrac{x^2}{2}$ and

$$\int x\log x dx = \frac{x^2}{2}\log x - \int \frac{x^2}{2}\frac{dx}{x\ln 10} = \frac{x^2}{2}\log x - \frac{1}{2\ln 10}\int xdx = \frac{x^2}{2}\log x - \frac{x^2}{4\ln 10} + C.$$

Example 5.4. Find $\displaystyle\int \arcsin x dx$.

Here we choose $u = \arcsin x$ and $dv = dx$ (actually here is no more possibilities!). Then $du = \dfrac{dx}{\sqrt{1-x^2}}$ and $v = \int dx = x$ and by (5.1)

$$\int \arcsin x dx = x\arcsin x - \int \frac{xdx}{\sqrt{1-x^2}}.$$

In the last integral we make the change of variable $t = 1-x^2$. Then $dt = -2xdx$ or $xdx = -\dfrac{dt}{2}$ and

$$\int \frac{xdx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} + C = -\sqrt{1-x^2} + C.$$

Thus,

$$\int \arcsin x dx = x\arcsin x + \sqrt{1-x^2} + C.$$

5.6 Integration of rational functions

Rational function is any function which can be defined by a rational fraction, i.e. an algebraic fraction such that both the numerator and the denominator are polynomials. It can be written in the form

$$f(x) = \frac{P_n(x)}{D_m(x)} \tag{6.1}$$

where $P_n(x)$ is the polynomial of degree n

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

and $D_m(x)$ is the polynomial of degree m

$$D_m(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_m x^m$$

For example the rational functions are

$$\frac{1}{x^2-1}, \quad \frac{2x^2-x+1}{x^3-x^2+x-1}, \quad \frac{x^3+1}{x^3-1}, \quad \frac{x^4}{x^2+1} \tag{6.2}$$

Rational function (6.1) is called *proper* if $n < m$, i.e. the degree of the numerator is less than the degree of the denominator. Rational function is called *improper* if $n \geq m$, i.e. the degree of the numerator is greater than, or equal to, the degree of the denominator. The *leading term* of the polynomial $P_n(x)$ is $a_n x^n$ and the leading term of the polynomial $D_m(x)$ is $b_m x^m$.

First two functions of (6.2) are proper rational functions and the third and fourth functions are improper rational functions.

5.6.1 Long division of polynomials

Improper rational function has to be performed as a sum of a polynomial and an proper rational function. This procedure is called long division of polynomials.

If $P_n(x)$ and $D_m(x)$ are polynomials, and the degree m of the denominator is less than, or equal to, the degree n of the numerator, then there exist unique polynomials $Q_{n-m}(x)$ and $R_k(x)$, so that

$$\frac{P_n(x)}{D_m(x)} = Q_{n-m}(x) + \frac{R_k(x)}{D_m(x)}$$

and so that the degree k of $R_k(x)$ is less than the degree m of $D_m(x)$. The polynomial $Q_{n-m}(x)$ is called the *quotient* and the polynomial $R_k(x)$ the *remainder*. In the special case where $R_k(x) = 0$, we say that $D_m(x)$ divides evenly into $P_n(x)$.

If the improper rational function is not complicated then the long division can be performed by simple elementary operations such as multiplying and division by the same number and adding and subtracting of the same quantity.

Example 6.1. Perform the long division of the improper rational function $\dfrac{x^2}{2x-1}$ and integrate the result.

First we multiply and divide this fraction by 4,

$$\frac{x^2}{2x-1} = \frac{1}{4}\frac{4x^2}{2x-1}$$

and then add to the numerator $-1+1$,

$$\frac{x^2}{2x-1} = \frac{1}{4}\frac{4x^2-1+1}{2x-1} = \frac{1}{4}\frac{4x^2-1}{2x-1} + \frac{1}{4}\frac{1}{2x-1}.$$

Canceling the first fraction gives

$$\frac{x^2}{2x-1} = \frac{1}{4}(2x+1) + \frac{1}{4(2x-1)}.$$

The result is the sum of the quotient, i.e. the polynomial $\dfrac{1}{4}(2x+1) = \dfrac{x}{2}+\dfrac{1}{4}$, and the proper rational function $\dfrac{1}{4(2x-1)}$. Now we can integrate the given rational function:

$$\int \frac{x^2dx}{2x-1} = \frac{1}{4}\int (2x+1)dx + \frac{1}{4}\int \frac{dx}{2x-1} = \frac{1}{4}(x^2+x) + \frac{1}{8}\ln|2x-1| + C$$

To find the last integral one can use the conclusion 4.6.

In more complicated cases the long division of polynomials works just like the long (numerical) division you did back in elementary school.

Example 6.2. Perform the long division of improper rational function

$$\frac{2x^4-3x^3+x^2-2}{x^2-3x+2}$$

The quotient of the leading terms of the dividend and the divisor is $\dfrac{2x^4}{x^2} = 2x^2$ and this is the first term of the quotient. We write it like the usual division of the numbers

$$\begin{array}{r|l} 2x^4-3x^3+x^2-2 & \underline{x^2-3x+2} \\ & 2x^2 \end{array}$$

Now we multiply the divisor $x^2 - 3x + 2$ by that $2x^2$ and write the answer $2x^4 - 6x^3 + 4x^2$ under the numerator polynomial, lining up terms of equal degree:

$$\begin{array}{l|l} 2x^4 - 3x^3 + x^2 - 2 & \underline{x^2 - 3x + 2} \\ 2x^4 - 6x^3 + 4x^2 & 2x^2 \end{array}$$

Next we subtract the last line from the line above it:

$$\begin{array}{r|l} 2x^4 - 3x^3 + x^2 - 2 & \underline{x^2 - 3x + 2} \\ \underline{2x^4 - 6x^3 + 4x^2} \quad & 2x^2 \\ 3x^3 - 3x^2 - 2 & \end{array}$$

Now we repeat the procedure: dividing the leading term $3x^3$ of the polynomial on the last line by the leading term x^2 of the divisor gives $3x$ and this is the second term of the quotient

$$\begin{array}{r|l} 2x^4 - 3x^3 + x^2 - 2 & \underline{x^2 - 3x + 2} \\ \underline{2x^4 - 6x^3 + 4x^2} \quad & 2x^2 + 3x \\ 3x^3 - 3x^2 - 2 & \end{array}$$

Now we multiply the divisor $x^2 - 3x + 2$ by $3x$ and write the answer $3x^3 - 9x^2 + 6x$ under the last line polynomial, lining up terms of equal degree. Then we subtract the line just written from the line above it:

$$\begin{array}{lll} 2x^4 - 3x^3 + x^2 - 2 & & |\underline{x^2 - 3x + 2} \\ \underline{2x^4 - 6x^3 + 4x^2} & & |2x^2 + 3x \\ \qquad 3x^3 - 3x^2 - 2 & & \\ \qquad \underline{3x^3 - 9x^2 + 6x} & & \\ \qquad\qquad 6x^2 - 6x - 2 & & \end{array}$$

We repeat this procedure once more: dividing the leading term $6x^2$ of the polynomial on the last line by the leading term x^2 of the divisor gives 6 and this is the third term of the quotient. Next we multiply the divisor x^2-3x+2 by 6 and write the answer $6x^2 - 18x + 12$ under the last line polynomial, lining up terms of equal degree. Then we subtract the line written from the line above it:

$$\begin{array}{lll} 2x^4 - 3x^3 + x^2 - 2 & & |\underline{x^2 - 3x + 2} \\ \underline{2x^4 - 6x^3 + 4x^2} & & |2x^2 + 3x + 6 \\ \qquad 3x^3 - 3x^2 - 2 & & \\ \qquad \underline{3x^3 - 9x^2 + 6x} & & \\ \qquad\qquad 6x^2 - 6x - 2 & & \\ \qquad\qquad \underline{6x^2 - 18x + 12} & & \\ \qquad\qquad\qquad 12x - 14 & & \end{array}$$

Now we have done it. The quotient is $2x^2+3x+6$ and the remainder $12x-14$, consequently

$$\frac{2x^4-3x^3+x^2-2}{x^2-3x+2}=2x^2+3x+6+\frac{12x-14}{x^2-3x+2}$$

The integration of the polynomial quotient is not a problem. Therefore we focus on the integration of the proper rational functions. To integrate the proper rational function, we have to decompose it into a sum of *partial fractions.*

5.6.2 Partial fractions

A general theorem in algebra states that every proper rational function can be expressed as a finite sum of fractions of the forms

$$\frac{A}{(x+a)^k} \quad \text{and} \quad \frac{Ax+B}{(ax^2+bx+c)^k}$$

where the integer $k \geq 1$ and A, B, a, b, c are constants with $b^2-4ac<0$. The condition $b^2-4ac<0$ means that the quadratic trinomial ax^2+bx+c cannot be factored into linear factors with real coefficients or, what amounts to the same thing, the quadratic equation $ax^2+bx+c=0$ has no real roots. Such a quadratic polynomial is said to be *irreducible.* When a rational function has been so expressed, we say that it has been decomposed into *partial fractions.* Therefore the problem of integration of this rational function reduces to that of integration of its partial fractions.

- The partial fraction of the first kind $\dfrac{A}{x+a}$;
- the partial fraction of the second kind $\dfrac{A}{(x+a)^k}$, where $k \in \mathbb{N}$ and $k>1$;
- the partial fraction of the third kind $\dfrac{Ax+B}{ax^2+bx+c}$, where the quadratic trinomial in the denominator is irreducible;
- the partial fraction of the fourth kind $\dfrac{Ax+B}{(ax^2+bx+c)^k}$, where $k \in \mathbb{N}$ and $k>1$ and the quadratic trinomial in the denominator is irreducible.

To integrate the partial fraction of the first kind, we use the equality of the differentials $dx=d(x+a)$ and the basic integral 2.2:

$$\int \frac{A}{x+a}dx = A\int \frac{d(x+a)}{x+a} = A\ln|x+a|+C. \tag{6.3}$$

Integrating the partial fraction of the second kind, we use again the equality of the differentials $dx = d(x+a)$ and the basic integral 2.1:

$$\int \frac{Adx}{(x+a)^k} = A\int (x+a)^{-k}d(x+a) = A\cdot\frac{(x+a)^{-k+1}}{-k+1}+C = -\frac{A}{(k-1)(x+a)^{k-1}}+C.$$

In general, the result of the integration of partial fraction of the third kind is the sum of natural logarithm and arc tangent. If in the numerator of the integrand there is only a constant, i.e. $A = 0$ then this integral equals to arc tangent.

Example 6.4. Find $\displaystyle\int \frac{dx}{9x^2+6x+5}$.

The quadratic polynomial $9x^2+6x+5 = 4+9x^2+6x+1 = 4+(3x+1)^2$ has no real roots. Using the differential $dx = \frac{1}{3}\cdot 3dx = \frac{1}{3}d(3x+1)$ and the basic integral 2.11 gives

$$\int \frac{dx}{9x^2+6x+5} = \frac{1}{3}\int \frac{d(3x+1)}{4+(3x+1)^2} = \frac{1}{3}\cdot\frac{1}{2}\arctan\frac{3x+1}{2}+C = \frac{1}{6}\arctan\frac{3x+1}{2}+C$$

If the numerator is the derivative of the denominator or a constant multiple of the derivative of the denominator then the result of the integration is natural logarithm.

Example 6.5. Find $\displaystyle\int \frac{(3x+1)dx}{9x^2+6x+5}$.

The derivative of the denominator $(9x^2+6x+5)' = 18x+6$ is 6 times the numerator and due to the equality $(3x+1)dx = \frac{1}{6}d(9x^2+6x+5)$ we have

$$\int \frac{(3x+1)dx}{9x^2+6x+5} = \frac{1}{6}\int \frac{d(9x^2+6x+5)}{9x^2+6x+5} = \frac{1}{6}\ln(9x^2+6x+5)+C.$$

In general, first we separate from the numerator the terms forming the derivative of the denominator. For this purpose we first multiply and divide the fraction by the same constant and next add to and subtract from the numerator the same constant. After doing that, the numerator of the second fraction is constant and the integral of that fraction is arc tangent.

Example 6.6. Find $\displaystyle\int \frac{(2x-1)dx}{9x^2+6x+5}$.

Using the results of the examples 6.4 and 6.5, we obtain

$$\int \frac{(2x-1)dx}{9x^2+6x+5} = \frac{1}{9}\int \frac{(18x-9)dx}{9x^2+6x+5} = \frac{1}{9}\int \frac{18x+6-6-9}{9x^2+6x+5}dx$$

$$= \frac{1}{9}\int \frac{(18x+6)dx}{9x^2+6x+5} - \frac{15}{9}\int \frac{dx}{9x^2+6x+5} = \frac{1}{9}\ln(9x^2+6x+5)$$

$$- \frac{5}{3}\cdot\frac{1}{6}\arctan\frac{3x+1}{2}+C = \frac{1}{9}\ln(9x^2+6x+5) - \frac{5}{18}\arctan\frac{3x+1}{2}+C.$$

5.6.3 Decomposition of rational function into a sum of partial fractions

To use the partial fractions for integration, we have first to decompose the proper rational function into a sum of partial fractions. The decomposition depends on the denominator: is the denominator the product of distinct linear factors, is the denominator the product of linear factors, some being repeated, or has the denominator the factors, which are the irreducible quadratic trinomials. Let us have three examples.

Example 6.7. Find the integral $\int \frac{(4x^2-3x-4)dx}{x^3+x^2-2x}$.

Factorizing the denominator gives

$$x^3+x^2-2x = x(x^2+x-2) = x(x-1)(x+2).$$

The denominator has three distinct linear factors or three distinct simple roots $x_1 = 0$, $x_2 = 1$ and $x_3 = -2$. For each factor we write one partial fraction of the first kind

$$\frac{4x^2-3x-4}{x^3+x^2-2x} \equiv \frac{4x^2-3x-4}{x(x-1)(x+2)} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}. \tag{6.4}$$

The coefficients A, B and C have to be determined so that the sum of these partial fractions is identical to the given rational function (i.e. equal for any value of x). Converting three partial fractions to the common denominator gives

$$\frac{4x^2-3x-4}{x(x-1)(x+2)} \equiv \frac{A(x-1)(x+2)+Bx(x+2)+Cx(x-1)}{x(x-1)(x+2)}.$$

If two fractions are identical and the denominators of these fractions are identical then the numerators are identical as well:

$$A(x-1)(x+2)+Bx(x+2)+Cx(x-1) \equiv 4x^2-3x-4. \tag{6.5}$$

The identity means that the above equality is true for every x. We select values for x which will make all but one of the coefficients go away. We will then be able to solve for that coefficient. More precisely,

if $x = 0$ then we obtain from (6.5) $-2A = -4$ or $A = 2$,
if $x = 1$ then we obtain from (6.5) $3B = -3$ or $B = -1$ and
if $x = -2$ then we obtain from (6.5) $6C = 18$ or $C = 3$.

Thus, we have determined the coefficients and using the partial fractions decomposition (6.4) gives

$$\int \frac{(4x^2-3x-4)dx}{x^3+x^2-2x} = \int \left(\frac{2}{x} - \frac{1}{x-1} + \frac{3}{x+2}\right) dx$$
$$= 2\int \frac{dx}{x} - \int \frac{dx}{x-1} + 3\int \frac{dx}{x+2} = 2\ln|x| - \ln|x-1| + 3\ln|x+2| + C.$$

Example 6.8. Find the integral $\int \frac{dx}{x^3+x^2-x-1}$.

Factorizing the denominator gives

$$x^3+x^2-x-1 = x^2(x+1)-(x+1) = (x+1)(x^2-1) = (x-1)(x+1)^2$$

The factor $x+1$ is two times repeated. The denominator has two roots, one simple root $x_1 = 1$ and one double root $x_2 = -1$. For the factor $x-1$ we write one partial fraction of the first kind (like in the previous example), for the two times repeated factor we write one partial factor of the first and one partial fraction of the second kind. Hence, the partial fraction decomposition is

$$\frac{1}{x^3+x^2-x-1} \equiv \frac{1}{(x-1)(x+1)^2} \equiv \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \qquad (6.6)$$

Generally, for the k times repeated factor we have to write k partial fractions: one partial fraction of the first kind and $k-1$ partial fractions of the second kind for each exponent from 2 up. Converting the right side of (6.6) to the common denominator, we obtain

$$\frac{1}{(x-1)(x+1)^2} \equiv \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2},$$

which yields the identity for the numerators

$$A(x+1)^2 + B(x-1)(x+1) + C(x-1) \equiv 1. \qquad (6.7)$$

Taking in (6.7) $x = 1$, we have $4A = 1$ i.e. $A = 1/4$. Taking in (6.7) $x = -1$ gives $-2C = 1$ or $C = -1/2$. There is no third root to determine the third coefficient. We take for x one random (possibly simple) value, for instance $x = 0$. The identity (6.7) yields $A - B - C = 1$. Using the values of A and C already found, we find $B = A - C - 1 = -1/4$. Now, by partial fractions decomposition (6.6) we find the integral

$$\int \frac{dx}{x^3+x^2-x-1} = \int \left(\frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+1} - \frac{1}{2}\frac{1}{(x+1)^2} \right) dx$$

$$= \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{(x+1)^2} = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}\cdot\frac{1}{x+1} + C$$

$$= \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2(x+1)} + C.$$

Example 6.9. Find the integral $\int \frac{x+1}{x^3+2x^2+3x}dx$.

Factorizing the denominator, we obtain

$$x^3 + 2x^2 + 3x = x(x^2 + 2x + 3)$$

The denominator has one real root $x_1 = 0$. The second factor is the irreducible quadratic trinomial. In the partial fractions decomposition we write for the simple root one partial fraction of the first kind and for the irreducible quadratic trinomial one partial fraction of the third kind. The partial fractions decomposition has to be the identity again:

$$\frac{x+1}{x^3 + 2x^2 + 3x} \equiv \frac{x+1}{x(x^2 + 2x + 3)} \equiv \frac{A}{x} + \frac{Bx + C}{x^2 + 2x + 3} \tag{6.8}$$

Taking the partial fractions to the common denominator gives

$$\frac{x+1}{x(x^2 + 2x + 3)} \equiv \frac{A(x^2 + 2x + 3) + (Bx + C)x}{x(x^2 + 2x + 3)},$$

which yields the identity of the numerators

$$Ax^2 + 2Ax + 3A + Bx^2 + Cx \equiv x + 1$$

In this case there is only one root, but we have to determine three coefficients. Therefore, we use the fact: if two polynomials are identically equal then the coefficients of the corresponding powers of x are equal. Converting this identity, we obtain

$$(A + B)x^2 + (2A + C)x + 3A \equiv x + 1$$

Equating the coefficients of the quadratic terms on the left side and on the right side (on the right side there is no quadratic term, i.e. the coefficient of this is zero) gives the equation $A + B = 0$. Equating the coefficients of the linear terms gives the equation $2A + C = 1$ and equating the constant terms gives the equation $3A = 1$. Thus, we have the system of linear equations

$$\left\{ \begin{array}{ccc} A + B & = & 0 \\ 2A + C & = & 1 \\ 3A & = & 1, \end{array} \right.$$

The solution of this system is $A = 1/3$, $B = -1/3$ and $C = 1/3$. Now, using the partial fractions decomposition (6.8), we find

$$\int \frac{x+1}{x^3+2x^2+3x}dx = \int \left(\frac{\frac{1}{3}}{x} + \frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+2x+3} \right) dx = \frac{1}{3}\int \frac{dx}{x} - \frac{1}{3}\int \frac{(x-1)dx}{x^2+2x+3}$$

$$= \frac{1}{3}\ln|x| - \frac{1}{3}\cdot\frac{1}{2}\int \frac{(2x-2)dx}{x^2+2x+3} = \frac{1}{3}\ln|x| - \frac{1}{6}\int \frac{(2x+2-4)dx}{x^2+2x+3}$$

$$= \frac{1}{6}\cdot 2\ln|x| - \frac{1}{6}\int \frac{(2x+2)dx}{x^2+2x+3} + \frac{4}{6}\int \frac{dx}{2+x^2+2x+1}$$

$$= \frac{1}{6}\ln x^2 - \frac{1}{6}\ln(x^2+2x+3) + \frac{2}{3}\int \frac{dx}{2+(x+1)^2}$$

$$= \frac{1}{6}\ln \frac{x^2}{x^2+2x+3} + \frac{2}{3\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} + C.$$

5.7 Integration of some classes of trigonometric functions

In this subsection we consider the integration of the rational functions with respect to trigonometric functions, i.e. the integrals

$$\int R(\sin x, \cos x)dx, \tag{7.1}$$

where $R(\sin x, \cos x)$ is the rational function with respect to $\sin x$ and $\cos x$. This kind of rational functions are for instance

$$\frac{1}{\sin x}, \quad \frac{1}{2+\cos x}, \quad \frac{\cos^3 x + \sin x}{\sin^2 x + \cos x}$$

or, in special case, the products like $\sin 2x \cos^2 x$.

5.7.1 General change of variable

In integral calculus, the tangent half-angle substitution is a substitution used for finding indefinite integrals of rational functions of trigonometric functions. The change of variable $t = \tan\frac{x}{2}$ always converts the integral (7.1) to the integral of a rational function. Indeed, first $\frac{x}{2} = \arctan t$ yields $x = 2\arctan t$, hence,

$$dx = \frac{2dt}{1+t^2}$$

Second

$$\sin x = \frac{\sin\left(2\cdot\frac{x}{2}\right)}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}$$

Dividing the numerator and the denominator of the last fraction by $\cos^2 \frac{x}{2}$, we obtain

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{2t}{1+t^2}$$

Third

$$\cos x = \frac{\cos\left(2 \cdot \frac{x}{2}\right)}{1} = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}}$$

or, dividing the numerator and the denominator by $\cos^2 \frac{x}{2}$, we have

$$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

Consequently, using the change of variable $t = \tan \frac{x}{2}$, we can convert the integral (7.1) to the integral of the rational function

$$\int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2dt}{1+t^2}$$

Example 7.1. Find the integral $\displaystyle\int \frac{dx}{2+\cos x}$.

We use the change of variable $t = \tan \frac{x}{2}$. Substituting $dx = \dfrac{2dt}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$, we obtain

$$\int \frac{dx}{2+\cos x} = \int \frac{\frac{2dt}{1+t^2}}{2 + \frac{1-t^2}{1+t^2}} = \int \frac{2dt}{3+t^2}$$

$$= 2 \cdot \frac{1}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}} + C = \frac{2}{\sqrt{3}} \arctan \frac{\tan \frac{x}{2}}{\sqrt{3}} + C.$$

5.7.2 Change of variable $t = \tan x$

The change of variable $t = \tan \frac{x}{2}$ is universal to integrate the expressions consisting of trigonometric functions. Sometimes it leads to the integration of rather complicated rational functions. This can be avoided if we use most straightforward methods. Let us consider two integrals

$$\int R(\sin^2 x, \cos^2 x) dx$$

and

$$\int R(\tan x) dx$$

The first integral is the rational function which contains only the even powers of sine and cosine functions, i.e. with respect to $\sin^2 x$ and $\cos^2 x$. The second integral is the rational function with respect to tangent function.

The change of variable $t = \tan x$ reduces both types of these integrals to the integral of rational function. With this change of variable, we get that $x = \arctan t$,

$$dx = \frac{dt}{1+t^2}, \tag{7.2}$$

$$\sin^2 x = \frac{\tan^2 x}{1+\tan^2 x} = \frac{t^2}{1+t^2} \tag{7.3}$$

and

$$\cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+t^2}. \tag{7.4}$$

Example 7.2. Find the integral $\displaystyle\int \frac{dx}{\cos 2x}$.

Using the double angle formula for cosine function and (7.2) - (7.4), we obtain

$$\int \frac{dx}{\cos 2x} = \int \frac{dx}{\cos^2 x - \sin^2 x} = \int \frac{\frac{dt}{1+t^2}}{\frac{1}{1+t^2} - \frac{t^2}{1+t^2}}$$

$$= \int \frac{dt}{1-t^2} = \frac{1}{2} \ln \left| \frac{1+t}{1-t} \right| + C = \frac{1}{2} \ln \left| \frac{1+\tan x}{1-\tan x} \right| + C$$

$$= \frac{1}{2} \ln \left| \frac{\cos x + \sin x}{\cos x - \sin x} \right| + C.$$

Example 7.3. Find the integral $\displaystyle\int \frac{\cos x dx}{\sin x + \cos x}$.

Dividing the numerator and the denominator by $\cos x$, we obtain the rational function with respect to $\tan x$. We substitute $t = \tan x$ and by (7.2) dx

$$\int \frac{\cos x dx}{\sin x + \cos x} = \int \frac{dx}{\tan x + 1} = \int \frac{\frac{dt}{1+t^2}}{t+1} = \int \frac{dt}{(1+t)(1+t^2)}$$

Using the partial fractions decomposition,

$$\frac{1}{(1+t)(1+t^2)} \equiv \frac{A}{1+t} + \frac{Bt+C}{1+t^2} \equiv \frac{A(1+t^2)+(Bt+C)(1+t)}{(1+t)(1+t^2)}$$

we obtain the identity for the numerators

$$A + At^2 + Bt + Bt^2 + C + Ct \equiv 1$$

or

$$(A+B)t^2 + (B+C)t + A + C \equiv 1$$

Equating the coefficients of the corresponding powers of t, we get the system of linear equations to determine the coefficients A, B and C (on the right hand side of this identity the coefficients of the square term and linear term are zeros)

$$\begin{cases} A+B=0 \\ B+C=0 \\ A+C=1, \end{cases}$$

The solution of this system of equations is $A = \frac{1}{2}, B = -\frac{1}{2}$ and $C = \frac{1}{2}$. Thus,

$$\int \frac{\cos x dx}{\sin x + \cos x} = \frac{1}{2}\int \frac{dt}{t+1} - \frac{1}{2}\int \frac{t-1}{t^2+1}dt = \frac{1}{2}\ln|t+1| - \frac{1}{4}\int \frac{2tdt}{t^2+1} + \frac{1}{2}\int \frac{dt}{t^2+1}$$
$$= \frac{1}{2}\ln|t+1| - \frac{1}{4}\ln\left(t^2+1\right) + \frac{1}{2}\arctan t + C.$$

Substituting $t = \tan x$ gives

$$\int \frac{\cos x dx}{\sin x + \cos x} = \frac{1}{2}\ln|\tan x + 1| - \frac{1}{4}\ln\left(\tan^2 x + 1\right) + \frac{1}{2}\arctan(\tan x) + C$$
$$= \frac{1}{2}\ln\left|\frac{\sin x + \cos x}{\cos x}\right| - \frac{1}{4}\ln\frac{1}{\cos^2 x} + \frac{1}{2}x + C$$
$$= \frac{1}{2}\ln\left|\frac{\sin x + \cos x}{\cos x}\right| + \frac{1}{2}\ln|\cos x| + \frac{1}{2}x + C = \frac{1}{2}\ln|\sin x + \cos x| + \frac{1}{2}x + C.$$

5.7.3 Change of variables $t = \sin x$ and $t = \cos x$

If the rational function is in form (or easily reducible to the form) $R(\sin x)\cos x$ then to find the integral

$$\int R(\sin x)\cos x dx \tag{7.5}$$

we use the change of variable $t = \sin x$, hence, $dt = \cos x dx$, and the integral (7.5) converts to the integral of the rational function $\int R(t)dt$.

Example 7.4. Find the integral $\displaystyle\int \frac{\sin 2x dx}{1 + \sin x}$.

The double angle formula for sine function gives

$$\int \frac{\sin 2x dx}{1 + \sin x} = \int \frac{2 \sin x}{1 + \sin x} \cos x dx$$

i.e. the integral (7.5). The substitution $t = \sin x$, $dt = \cos x dx$ gives

$$\int \frac{\sin 2x dx}{1 + \sin x} = 2 \int \frac{t dt}{1 + t} = 2 \int \frac{t + 1 - 1}{1 + t} = 2 \int \frac{t + 1}{1 + t} dt - 2 \int \frac{dt}{1 + t}$$

$$= 2 \int dt - 2 \int \frac{dt}{1 + t} = 2t - 2 \ln |1 + t| + C = 2 \sin x - 2 \ln |1 + \sin x| + C.$$

To find the integral

$$\int R(\cos x) \sin x dx \tag{7.6}$$

we change the variable $t = \cos x$. Then $dt = -\sin x dx$ and the integral (7.6) converts to the integral of the rational function $-\int R(t)dt$.

Example 7.5. Find the integral $\displaystyle\int \frac{\sin x dx}{\cos x - \cos^2 x}$. This integral is in the form (7.6). By change of variable $t = \cos x$, $dt = -\sin x dx$ we obtain that

$$\int \frac{\sin x dx}{\cos x - \cos^2 x} = - \int \frac{dt}{t - t^2} = \int \frac{dt}{t^2 - t}$$

Now the integral has been converted to the integral of the rational function we use the partial fractions decomposition

$$\frac{1}{t^2 - t} \equiv \frac{1}{t(t - 1)} \equiv \frac{A}{t} + \frac{B}{t - 1} \equiv \frac{A(t - 1) + Bt}{t(t - 1)}$$

This yields the identity of the numerators

$$A(t - 1) + Bt \equiv 1$$

Taking $t = 1$ gives $B = 1$ and taking $t = 0$ gives $A = -1$. Thus,

$$\int \frac{\sin x dx}{\cos x - \cos^2 x} = - \int \frac{dt}{t} + \int \frac{dt}{t - 1}$$

$$= \ln |t - 1| - \ln |t| + C = \ln \left| \frac{t - 1}{t} \right| + C = \ln \left| \frac{\cos x - 1}{\cos x} \right| + C.$$

5.7.4 More techniques for integration of trigonometric expressions

The products of the even powers of sine and cosine functions, i.e. the integral

$$\int \sin^{2n} x \cos^{2m} x dx \tag{7.7}$$

can be integrated, using the sine of half-angle and cosine of half-angle formulas

$$\sin^2 x = \frac{1-\cos 2x}{2} \tag{7.8}$$

and

$$\cos^2 x = \frac{1+\cos 2x}{2} \tag{7.9}$$

Example 7.6. Find the integral $\int \sin^4 x \cos^2 x dx$.

By formulas (7.8) and (7.9) we have

$$\int \sin^4 x \cos^2 x dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 \frac{1+\cos 2x}{2} dx$$

$$= \frac{1}{8}\int (1 - 2\cos 2x + \cos^2 2x)(1+\cos 2x)dx = \frac{1}{8}\int (1-\cos 2x - \cos^2 2x + \cos^3 2x)dx$$

$$= \frac{1}{8}\int \left[\sin^2 2x - \cos 2x(1-\cos^2 2x)\right] dx = \frac{1}{8}\int \frac{1-\cos 4x}{2} dx - \frac{1}{8}\int \sin^2 2x \cos 2x dx.$$

The second integral is in the form (7.5). Changing the variable $t = \sin 2x$, we obtain that $dt = 2\cos 2x dx$ or $\cos 2x dx = \frac{1}{2} dt$ and

$$\int \sin^4 x \cos^2 x dx = \frac{1}{16}\int (1-\cos 4x)\, dx - \frac{1}{16}\int t^2 dt$$

$$= \frac{x}{16} - \frac{1}{64}\sin 4x - \frac{t^3}{48} + C = \frac{x}{16} - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C$$

$$= \frac{x}{16} - \frac{1}{16}\sin x \cos^3 x + \frac{1}{16}\sin^3 x \cos x - \frac{1}{6}\sin^3 x \cos^3 x + C.$$

The last transformation is not obvious, the reader has to check it oneself.

To integrate the products $\sin ax \cos bx$, $\cos ax \cos bx$ and $\sin ax \sin bx$, we use the product-to-sum formulas of the sine and cosine functions

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)], \tag{7.10}$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \tag{7.11}$$

and

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]. \tag{7.12}$$

Example 7.7. Find the integral $\displaystyle\int \sin 5x\cos 4x\sin 3x dx$.

By the formula (7.10) we obtain

$$\begin{aligned}\int \sin 5x\cos 4x\sin 3x dx &= \frac{1}{2}\int(\sin 9x + \sin x)\sin 3x dx\\ &= \frac{1}{2}\int \sin 9x\sin 3x dx + \frac{1}{2}\int \sin x\sin 3x dx\end{aligned}$$

and by the formula (7.12)

$$\begin{aligned}\int \sin 5x\cos 4x\sin 3x dx &= \frac{1}{4}\int(\cos 6x - \cos 12x)dx + \frac{1}{4}\int(\cos 2x - \cos 4x)dx\\ &= \frac{1}{4}\left(\frac{1}{6}\sin 6x - \frac{1}{12}\sin 12x\right) + \frac{1}{4}\left(\frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right) + C\\ &= \frac{1}{24}\sin 6x - \frac{1}{48}\sin 12x + \frac{1}{8}\sin 2x - \frac{1}{16}\sin 4x + C.\end{aligned}$$

5.8 Integration of rational functions with respect to e^x

Let $R(e^x)$ denotes the rational function with respect to the exponential function e^x. To find the integral $\int R(e^x)dx$, we use the change of variable $t = e^x$. Then $dt = e^x dx$, i.e. $dt = t dx$ and it follows that $dx = \dfrac{dt}{t}$.

Example 8.1. Find the integral $\displaystyle\int \frac{e^{2x} - 2e^x}{e^{2x}+1}dx$.

By change of the variable $t = e^x$ we have $dx = \dfrac{dt}{t}$ and

$$\begin{aligned}\int \frac{e^{2x} - 2e^x}{e^{2x}+1}dx &= \int \frac{t^2 - 2t}{t^2+1}\cdot\frac{dt}{t} = \int \frac{t-2}{t^2+1}dt = \frac{1}{2}\int \frac{2t}{t^2+1}dt - 2\int \frac{dt}{t^2+1}\\ &= \frac{1}{2}\ln(t^2+1) - 2\arctan t + C = \frac{1}{2}\ln(e^{2x}+1) - 2\arctan e^x + C.\end{aligned}$$

5.9 Integrals of irrational functions

Irrational functions are the functions containing radicals, for instance

$$\frac{\sqrt[3]{x-1}}{1+\sqrt{x-1}} \quad \text{or} \quad x^2\sqrt{4-x^2}$$

The general principle of the integration of the irrational functions is similar to the integration of trigonometric functions. We are looking for the change of variable, which converts the integral of the irrational function to the integral of the rational function.

5.9.1 Integration by power substitution

Let us consider the integral

$$\int R\left(x, \sqrt[m]{\frac{ax+b}{cx+d}}, ..., \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx. \tag{9.1}$$

where the integrand contains the variable x and different roots of the linear fractional function $\frac{ax+b}{cx+d}$, where a, b, c and d are constants. The integral is convertible to the integral of the rational function, using the change of variable

$$\frac{ax+b}{cx+d} = t^k, \tag{9.2}$$

where k is the least common multiple of the indexes of the roots m, ..., n. We solve the equation (9.2) for x and find the differential dx.

Example 9.1. Find the integral $\int \frac{\sqrt[3]{x-1}}{1+\sqrt{x-1}} dx$.

The integrand contains two roots of $x-1$. The least common multiple of the indexes of the roots is $2 \cdot 3 = 6$ and therefore we use the change of variable (9.2) $x-1=t^6$. It follows that $x = t^6+1$, $dx = 6t^5dt$, $\sqrt[3]{x-1} = t^2$, $\sqrt{x-1} = t^3$ and (from the point of view of the later re-substitution) $t = \sqrt[6]{x-1}$.

After substitutions, we obtain the integral of the rational function

$$\int \frac{\sqrt[3]{x-1}}{1+\sqrt{x-1}} dx = \int \frac{t^2 \cdot 6t^5 dt}{1+t^3} = 6 \int \frac{t^7 dt}{1+t^3}$$

The integrand is the improper rational function and first we have to divide the polynomials. The result of the division is

$$\frac{t^7 dt}{1+t^3} = t^4 - t + \frac{t}{1+t^3}$$

For the integration of the proper rational fraction we use the partial fractions decomposition

$$\frac{t}{1+t^3} \equiv \frac{t}{(1+t)(1-t+t^2)} \equiv \frac{A}{1+t} + \frac{Bt+C}{1-t+t^2} \equiv \frac{A(1-t+t^2)+(Bt+C)(1+t)}{(1+t)(1-t+t^2)}$$

It follows the identity of the numerators

$$(A+B)t^2+(B+C-A)t+A+C\equiv t$$

and, equating the coefficients of the corresponding powers of t gives the system of linear equations

$$\begin{cases} A+B=0 \\ B+C-A=1 \\ A+C=0. \end{cases}$$

The solutions of this system are $A=-\dfrac{1}{3}$, $B=\dfrac{1}{3}$ and $C=\dfrac{1}{3}$. Now, using the partial fractions decomposition, we find

$$\begin{aligned}
&\int \frac{\sqrt[3]{x-1}}{1+\sqrt{x-1}}dx = 6\int (t^4-t)dt + 6\int \frac{tdt}{1+t^3} \\
=\ & \frac{6t^5}{5}-\frac{6t^2}{2}+6\cdot\left(-\frac{1}{3}\right)\int\frac{dt}{t+1}+6\cdot\frac{1}{3}\int\frac{(t+1)dt}{t^2-t+1} \\
=\ & \frac{6t^5}{5}-3t^2-2\ln|t+1|+\int\frac{2t+2}{t^2-t+1}dt=\frac{6t^5}{5}-3t^2-2\ln|t+1|+\int\frac{2t-1+3}{t^2-t+1}dt \\
=\ & \frac{6t^5}{5}-3t^2-2\ln|t+1|+\int\frac{2t-1}{t^2-t+1}dt+3\int\frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}} \\
=\ & \frac{6t^5}{5}-3t^2-2\ln|t+1|+\ln(t^2-t+1)+3\cdot\frac{2}{\sqrt{3}}\arctan\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}+C \\
=\ & \frac{6t^5}{5}-3t^2+\ln\frac{t^2-t+1}{(t+1)^2}+2\sqrt{3}\arctan\frac{2t-1}{\sqrt{3}}+C \\
=\ & \frac{6\sqrt[6]{(x-1)^5}}{5}-3\sqrt[3]{x-1}+\ln\frac{\sqrt[3]{x-1}-\sqrt[6]{x-1}+1}{\left(\sqrt[6]{x-1}+1\right)^2}+2\sqrt{3}\arctan\frac{2\sqrt[6]{x-1}-1}{\sqrt{3}}+C.
\end{aligned}$$

5.9.2 Integrals of irrational functions. Trigonometric substitutions

Let us consider the integral

$$\int R\left(x,\sqrt{ax^2+bx+c}\right)dx. \tag{9.3}$$

It is always possible to separate the square of binomial from the quadratic trinomial ax^2+bx+c under the radical sign of (9.3). We have to convert this quadratic trinomial as follows

$$\begin{aligned} ax^2+bx+c &= a\left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right]+c-\frac{b^2}{4a} \\ &= a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a} \end{aligned}$$

Denoting by $k=\dfrac{4ac-b^2}{4a}$ and using the linear change of the variable $t=x+\dfrac{b}{2a}$, $dt=dx$, the integral (9.3) converts to

$$\int R(t,\sqrt{at^2+k})dt$$

Denoting in the last integral the variable of integration by x again, we have

$$\int R(x,\sqrt{ax^2+k})dx \tag{9.4}$$

In (9.4) there are three possibilities depending on the signs of the coefficient a and constant k.

First, let a and k be positive. Then we can write the integral (9.4) as

$$\int R(x,\sqrt{a^2x^2+k^2})dx$$

The irrationality is removable by the change of variable

$$x=\frac{k}{a}\tan t \tag{9.5}$$

because in this case

$$dx=\frac{kdt}{a\cos^2 t}$$

and

$$\sqrt{a^2x^2+k^2}=\sqrt{k^2\tan^2 t+k^2}=\frac{k}{\cos t}$$

Example 9.2. Find the integral $\displaystyle\int\frac{dx}{x^2\sqrt{x^2+2}}$.

Changing the variable (9.5) $x=\sqrt{2}\tan t$, we obtain that

$$dx=\frac{\sqrt{2}dt}{\cos^2 t},\quad \sqrt{x^2+2}=\sqrt{2\tan^2 t+2}=\frac{\sqrt{2}}{\cos t}$$

and

$$\int \frac{dx}{x^2\sqrt{x^2+2}} = \int \frac{\sqrt{2}dt}{2\tan^2 t \dfrac{\sqrt{2}}{\cos t}\cos^2 t} = \frac{1}{2}\int \frac{\cos t dt}{\sin^2 t}$$

To find the last integral, we substitute $z = \sin t$. Then $dz = \cos t dt$ and

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2+2}} &= \frac{1}{2}\int \frac{dz}{z^2} = -\frac{1}{2z} + C = -\frac{1}{2\sin t} + C \\
&= -\frac{1}{\frac{2\tan t}{\sqrt{1+\tan^2 t}}} + C = -\frac{\sqrt{1+\tan^2 t}}{2\tan t} + C = -\frac{\sqrt{1+\frac{x^2}{2}}}{2\cdot\frac{x}{\sqrt{2}}} + C \\
&= -\frac{\sqrt{2+x^2}}{2x} + C.
\end{aligned}$$

Second, let in (9.4) $a < 0$ and $k > 0$. In this case (9.4) can be written

$$\int R(x, \sqrt{k^2 - a^2x^2})dx$$

The irrationality is removable, using the change of variable

$$x = \frac{k}{a}\sin t, \tag{9.6}$$

for which

$$dx = \frac{k}{a}\cos t dt$$

and

$$\sqrt{k^2 - a^2x^2} = \sqrt{k^2 - k^2\sin^2 t} = k\cos t$$

Example 9.3. Find the integral $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}dx$.

By the change of variable (9.6) $x = \sin t$ we have $dx = \cos t dt$, $\sqrt{1-x^2} = \cos t$ and

$$\begin{aligned}
\int \frac{\sqrt{1-x^2}}{x^2}dx &= \int \frac{\cos t \cos t dt}{\sin^2 t} = \int \frac{1-\sin^2 t}{\sin^2 t}dt = \int \frac{dt}{\sin^2 t} - \int dt \\
&= -\cot t - t + C = -\frac{\sqrt{1-\sin^2 t}}{\sin t} - t + C = -\frac{\sqrt{1-x^2}}{x} - \arcsin x + C.
\end{aligned}$$

Third, let $a > 0$ and $k < 0$. Then (9.4) can be written as

$$\int R(x, \sqrt{a^2x^2 - k^2})dx$$

The irrationality is removable, using the change of variable

$$x = \frac{k}{a}\cosh t, \tag{9.7}$$

for which

$$dx = \frac{k}{a}\sinh t dt$$

and

$$\sqrt{a^2x^2 - k^2} = \sqrt{k^2\cosh^2 t - k^2} = \sqrt{k^2\sinh^2 t} = k\sinh t$$

Example 9.4. Find the integral $\displaystyle\int \frac{\sqrt{x^2-4}}{x^2}dx$.

By the change of variable (9.7) $x = 2\cosh t$, we obtain that

$$dx = 2\sinh t dt, \quad \sqrt{x^2-4} = 2\sinh t$$

and

$$\int \frac{\sqrt{x^2-4}}{x^2}dx = \int \frac{2\sinh t \cdot 2\sinh t dt}{4\cosh^2 t} = \int \frac{4\sinh^2 t}{4\cosh^2 t}dt$$

$$= \int \frac{\cosh^2 t - 1}{\cosh^2 t}dt = \int dt - \int \frac{dt}{\cosh^2 t} = t - \tanh t + C$$

$$= \operatorname{arcosh}\frac{x}{2} - \frac{\sinh t}{\cosh t} + C = \operatorname{arcosh}\frac{x}{2} - \frac{\sqrt{\cosh^2 t - 1}}{\cosh t} + C$$

$$= \ln\left|\frac{x}{2} + \sqrt{\frac{x^2}{4} - 1}\right| - \frac{\sqrt{\frac{x^2}{4} - 1}}{\frac{x}{2}} + C$$

$$= \ln\left|\frac{x + \sqrt{x^2-4}}{2}\right| - \frac{\sqrt{x^2-4}}{x} + C = \ln|x + \sqrt{x^2-4}| - \frac{\sqrt{x^2-4}}{x} + C$$

Here the whatever constant $-\ln 2 + C$ has been replaced by C again.

5.10 Exercises

Integration by the table of integrals

1. $\displaystyle\int (x + \sqrt{x})\,dx.$ Answer: $\displaystyle\frac{x^2}{2} + \frac{2x\sqrt{x}}{3} + C.$

2. $\displaystyle\int \left(\frac{1}{x^2} + \frac{4}{x\sqrt{x}} + 2\right)dx.$ Answer: $\displaystyle-\frac{1}{x} - \frac{8}{\sqrt{x}} + 2x + C.$

3. $\displaystyle\int \frac{(1-x)^2}{x\sqrt{x}}dx.$ Answer: $\displaystyle\frac{2}{3}x\sqrt{x} - 4\sqrt{x} - \frac{2}{\sqrt{x}} + C.$

4. $\int (\sqrt{x}-1)(x+\sqrt{x}+1)dx$ Answer: $\frac{2x^2\sqrt{x}}{5} - x + C.$

5. $\int \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}}dx.$ Answer: $\frac{6}{7}x\sqrt[6]{x} - \frac{4}{3}\sqrt[4]{x^3} + C.$

6. $\int \cot^2 x dx.$ Answer: $-\cot x - x + C.$

7. $\int \frac{3\cdot 2^x - 2\cdot 3^x}{2^x}dx.$ Answer: $3x - \frac{2\cdot(1,5)^x}{\ln 1,5} + C.$

8. $\int \frac{1+\cos^2 x}{1+\cos 2x}dx.$ Answer: $\frac{1}{2}(\tan x + x) + C.$

9. $\int \frac{(1+2x^2)dx}{x^2(1+x^2)}.$ Answer: $\arctan x - \frac{1}{x} + C.$

10. $\int \frac{dx}{\sqrt{3-3x^2}}.$ Answer: $\frac{1}{\sqrt{3}}\arcsin x + C.$

11. $\int \frac{dx}{\sqrt{4+x^2}}.$ Answer: $\ln\left(x+\sqrt{4+x^2}\right) + C.$

12. $\int \frac{dx}{x^2-5}.$ Answer: $\frac{1}{2\sqrt{5}}\ln\left|\frac{\sqrt{5}-x}{\sqrt{5}+x}\right| + C.$

13. $\int \tanh^2 x dx.$ Answer: $x - \tanh x + C.$

Integration by the change of variable

14. $\int (x+1)^{11}dx.$ Answer: $\frac{(x+1)^{12}}{12} + C.$

15. $\int \sqrt{5-2x}dx.$ Answer: $-\frac{(5-2x)\sqrt{5-2x}}{3} + C.$

16. $\int x\sqrt{x^2+1}dx.$ Answer: $\frac{1}{3}(x^2+1)\sqrt{x^2+1} + C.$

17. $\int \frac{x^3dx}{\sqrt{x^4+3}}.$ Answer: $\frac{1}{2}\sqrt{x^4+3} + C.$

18. $\int \cos(5x-2)dx.$ Answer: $\frac{1}{5}\sin(5x-2) + C.$

19. $\int \tan x dx.$ Answer: $-\ln|\cos x| + C.$

20. $\int \sin^4 x \cos x dx.$ Answer: $\frac{1}{5}\sin^5 x + C.$

21. $\int \frac{\sin 2x}{1+\cos^2 x} dx.$ Answer: $-\ln(1+\cos^2 x) + C.$

22. $\int \frac{\tan x}{\cos^2 x} dx.$ Answer: $\frac{\tan^2 x}{2} + C.$

23. $\int \frac{dx}{\sin^2 x\sqrt{1+\cot x}}.$ Answer: $-2\sqrt{1+\cot x} + C.$

24. $\int \frac{e^x dx}{e^x+2}.$ Answer: $\ln(e^x+2) + C.$

25. $\int xe^{-x^2} dx.$ Answer: $-\frac{1}{2}e^{-x^2} + C.$

26. $\int \cos x e^{\sin x} dx.$ Answer: $e^{\sin x} + C.$

27. $\int 2^{3x-1} dx.$ Answer: $\frac{2^{3x-1}}{3\ln 2} + C.$

28. $\int \frac{\ln x}{x} dx.$ Answer: $\frac{\ln^2 x}{2} + C.$

29. $\int \frac{dx}{x\ln x}.$ Answer: $\ln|\ln x| + C.$

30. $\int \frac{dx}{1+4x^2}.$ Answer: $\frac{1}{2}\arctan 2x + C.$

31. $\int \frac{dx}{\sqrt{4-9x^2}}.$ Answer: $\frac{1}{3}\arcsin\frac{3x}{2} + C.$

32. $\int \frac{xdx}{x^4+1}.$ Answer: $\frac{1}{2}\arctan x^2 + C.$

33. $\int \frac{e^x dx}{e^{2x}+1}.$ Answer: $\arctan e^x + C.$

34. $\int \frac{e^x dx}{\sqrt{1-e^{2x}}}.$ Answer: $\arcsin e^x + C.$

35. $\displaystyle\int \frac{dx}{(1+x^2)\arctan x}$. Answer: $\ln|\arctan x| + C$.

36. $\displaystyle\int \frac{dx}{x\sqrt{1-\ln^2 x}}$. Answer: $\arcsin(\ln x) + C$.

37. $\displaystyle\int \frac{e^{2x}-1}{e^x}dx$. Answer: $e^x + e^{-x} + C$.

38. $\displaystyle\int \frac{1+x}{\sqrt{1-x^2}}dx$. Answer: $\arcsin x - \sqrt{1-x^2} + C$.

39. $\displaystyle\int \frac{3x-1}{x^2+4}dx$. Answer: $\displaystyle\frac{3}{2}\ln(x^2+4) - \frac{1}{2}\arctan\frac{x}{2} + C$.

40. $\displaystyle\int \frac{2x-\sqrt{\arcsin x}}{\sqrt{1-x^2}}dx$. Answer: $\displaystyle -2\sqrt{1-x^2} - \frac{2\arcsin x\sqrt{\arcsin x}}{3} + C$.

41. $\displaystyle\int \frac{\cosh x}{1+\sinh x}dx$. Answer: $\ln|1+\sinh x| + C$.

42. $\displaystyle\int \frac{dx}{\sinh x\cosh x}$. Answer: $\ln|\tanh x| + C$.

43. $\int \sinh^3 x dx$. Answer: $\displaystyle\frac{\cosh^3 x}{3} - \cosh x + C$.

Integration by parts

44. $\displaystyle\int xe^{-x}dx$. Answer: $-xe^{-x} - e^{-x} + C$.

45. $\displaystyle\int (x+2)\sin 2x dx$. Answer: $\displaystyle -\frac{x+2}{2}\cos 2x + \frac{1}{4}\sin 2x + C$.

46. $\displaystyle\int x\cos\frac{x}{2}dx$. Answer: $\displaystyle 2x\sin\frac{x}{2} + 4\cos\frac{x}{2} + C$.

47. $\displaystyle\int x3^x dx$. Answer: $\displaystyle\frac{x\cdot 3^x}{\ln 3} - \frac{3^x}{\ln^2 3} + C$.

48. $\displaystyle\int x\sinh x dx$. Answer: $x\cosh x - \sinh x + C$

49. $\displaystyle\int \ln x dx$. Answer: $x\ln x - x + C$.

50. $\displaystyle\int \arccos x dx.$ Answer: $x \arccos x - \sqrt{1-x^2} + C.$

51. $\displaystyle\int x \arctan x dx.$ Answer: $\dfrac{1}{2}\left[(x^2+1)\arctan x - x\right] + C.$

52. $\displaystyle\int \ln(x^2+1) dx.$ Answer: $x\ln(x^2+1) - 2x + 2\arctan x + C.$

53. $\displaystyle\int \arctan\sqrt{x} dx.$ Answer: $(x+1)\arctan\sqrt{x} - \sqrt{x} + C.$

54. $\displaystyle\int x^2 \ln(1+x) dx$ Answer: $\dfrac{(x^3+1)\ln(1+x)}{3} - \dfrac{x^3}{9} + \dfrac{x^2}{6} - \dfrac{x}{3} + C.$

55. $\displaystyle\int \frac{\arcsin\sqrt{x}}{\sqrt{x}} dx.$ Answer: $2\sqrt{x}\arcsin\sqrt{x} + 2\sqrt{1-x} + C.$

56. $\displaystyle\int x \tan^2 x dx.$ Answer: $x\tan x - \dfrac{x^2}{2} + \ln|\cos x| + C.$

Division of the polynomials

57. $\displaystyle\int \frac{x}{2x+1} dx.$ Answer: $\dfrac{x}{2} - \dfrac{1}{4}\ln|2x+1| + C.$

58. $\displaystyle\int \frac{2x+3}{3x+2} dx.$ Answer: $\dfrac{2}{3}x + \dfrac{5}{9}\ln|3x+2| + C.$

59. $\displaystyle\int \frac{(1+x)^2}{x^2+1} dx.$ Answer: $x + \ln(x^2+1) + C.$

60. $\displaystyle\int \frac{x^2-1}{x^2+1} dx.$ Answer: $x - 2\arctan x + C.$

61. $\displaystyle\int \frac{x^3}{x+1} dx.$ Answer: $\dfrac{x^3}{3} - \dfrac{x^2}{2} + x - \ln|x+1| + C.$

62. $\displaystyle\int \frac{x^2 dx}{9-x^2}.$ Answer: $\dfrac{3}{2}\ln\left|\dfrac{3+x}{3-x}\right| - x + C.$

63. $\displaystyle\int \frac{x^5}{x^3-1} dx.$ Answer: $\dfrac{x^3}{3} + \dfrac{1}{3}\ln|x^3-1| + C.$

Integration of the rational functions

64. $\displaystyle\int \frac{2x-1}{x^2-3x+2} dx.$ Answer: $3\ln|x-2| - \ln|x-1| + C.$

65. $\int \frac{3x+2}{x^2+x}dx.$ Answer: $2\ln|x| + \ln|x+1| + C.$

66. $\int \frac{x^2+2x+6}{(x-1)(x-2)(x-4)}dx.$ Answer: $3\ln|x-1| - 7\ln|x-2| + 5\ln|x-4| + C.$

67. $\int \frac{dx}{6x^3-7x^2-3x}.$ Answer: $\frac{2}{33}\ln|2x-3| + \frac{3}{11}\ln|3x+1| - \frac{1}{3}\ln|x| + C.$

68. $\int \frac{x^2+1}{(x^2-1)(x^2-4)}dx.$ Answer: $\frac{1}{3}\ln\left|\frac{x+1}{x-1}\right| + \frac{5}{12}\ln\left|\frac{x-2}{x+2}\right| + C.$

69. $\int \frac{x^5+x^4-8}{x^3-4x}dx.$ Answer: $\frac{x^3}{3} + \frac{x^2}{2} + 4x + 2\ln|x| + 5\ln|x-2| - 3\ln|x+2| + C.$

70. $\int \frac{dx}{x(x-1)^2}.$ Answer: $\ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C.$

71. $\int \frac{x-8}{x^3-4x^2+4x}dx.$ Answer: $\frac{3}{x-2} + \ln\left(\frac{x-2}{x}\right)^2 + C.$

72. $\int \frac{2x-1}{x^2(x-1)^2}dx.$ Answer: $\frac{1}{x} - \frac{1}{x-1} + C.$

73. $\int \frac{3x-2}{x(x^2+1)}dx.$ Answer: $\ln\frac{x^2+1}{x^2} + 3\arctan x + C.$

74. $\int \frac{dx}{x(x^2+2x+2)}.$ Answer: $\frac{1}{2}\ln|x| - \frac{1}{4}\ln(x^2+2x+2) - \frac{1}{2}\arctan(x+1) + C.$

75. $\int \frac{dx}{x^3+1}$ Answer: $\frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C.$

76. $\int \frac{3x^2+5x+12}{x^4+4x^2+3}dx.$ Answer: $\frac{5}{4}\ln(x^2+1) - \frac{5}{4}\ln(x^2+3) + \frac{9}{2}\arctan x - \frac{\sqrt{3}}{2}\arctan\frac{x}{\sqrt{3}} + C.$

Integration of the trigonometric functions

77. $\int \frac{dx}{4+5\cos x}.$ Answer: $\frac{1}{3}\ln\left|\frac{\tan\frac{x}{2}+3}{\tan\frac{x}{2}-3}\right| + C.$

78. $\displaystyle\int \frac{dx}{(1+\cos x)\sin x}$. Answer: $\displaystyle\frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + \frac{1}{4}\tan^2\frac{x}{2} + C.$

79. $\displaystyle\int \frac{dx}{\sin x + \cos x}$. Answer:$\displaystyle\frac{\sqrt{2}}{2}\ln\left|\tan\left(\frac{\pi}{8} + \frac{x}{2}\right)\right| + C.$

80. $\displaystyle\int \frac{dx}{5 - 4\sin x + 3\cos x}$. Answer: $\displaystyle\frac{1}{2 - \tan\frac{x}{2}} + C.$

81. $\displaystyle\int \frac{dx}{\sin^3 x}$. Answer: $\displaystyle\frac{1}{8}\tan^2\frac{x}{2} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{8}\cot^2\frac{x}{2} + C.$

82. $\displaystyle\int \frac{\tan x}{1 - 2\tan x}dx$. Answer: $\displaystyle-\frac{1}{5}\ln|2\sin x - \cos x| - \frac{2}{5}x + C.$

83. $\displaystyle\int \frac{dx}{\tan x \cos 2x}$. Answer: $\displaystyle\ln\frac{|\sin x|}{\sqrt{\cos 2x}} + C.$

84. $\displaystyle\int \frac{dx}{\cos^6 x}$. Answer: $\displaystyle\tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C.$

85. $\displaystyle\int \frac{dx}{1 + \sin^2 x}$. Answer: $\displaystyle\frac{\sqrt{2}}{2}\arctan(\sqrt{2}\tan x) + C.$

86. $\displaystyle\int \frac{\sin x dx}{(1 - \cos x)^2}$. Answer: $\displaystyle\frac{1}{\cos x - 1} + C.$

87. $\displaystyle\int \frac{\sin^3 x dx}{\cos^4 x}$. Answer: $\displaystyle\frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C.$

88. $\displaystyle\int \frac{\tan x dx}{1 + \cos x}$. Answer: $\displaystyle\ln\left|\frac{1 + \cos x}{\cos x}\right| + C.$

89. $\displaystyle\int \frac{\sin^3 x dx}{\cos^2 x + 1}$. Answer: $\cos x - 2\arctan(\cos x) + C.$

90. $\displaystyle\int \sin^2 x dx$. Answer: $\displaystyle\frac{1}{2}x - \frac{1}{4}\sin 2x + C.$

91. $\displaystyle\int \cos^4 x dx$. Answer: $\displaystyle\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$

92. $\displaystyle\int \sin^2 x \cos^2 x dx$. Answer: $\displaystyle\frac{1}{8}x - \frac{1}{32}\sin 4x + C.$

93. $\displaystyle\int \sin^6 x dx$. Answer: $\displaystyle\frac{5}{16}x - \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$

94. $\int \sin 5x \sin 3x dx.$ Answer: $\frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C.$

95. $\int \cos 2x \cos x dx.$ Answer: $\frac{1}{2}\sin x + \frac{1}{6}\sin 3x + C.$

96. $\int \sin 4x \cos 3x dx.$ Answer: $-\frac{1}{14}\cos 7x - \frac{1}{2}\cos x + C.$

Integration of the rational functions with respect to e^x

97. $\int \frac{e^x dx}{e^{2x}+1}.$ Answer: $\arctan e^x + C.$

98. $\int \frac{e^{2x} dx}{e^x+1}.$ Answer: $e^x - \ln(e^x+1) + C.$

99. $\int \frac{dx}{e^x+e^{2x}}.$ Answer: $\ln(1+e^x) - e^{-x} - x + C.$

100. $\int \frac{e^x dx}{e^{2x}-6e^x+13}.$ Answer: $\frac{1}{2}\arctan\frac{e^x-3}{2} + C.$

Integration of the irrational functions

101. $\int \frac{\sqrt{x}}{x(x+1)}dx.$ Answer: $2\arctan\sqrt{x} + C.$

102. $\int \frac{2+x}{\sqrt[3]{3-x}}dx.$ Answer: $\frac{3}{5}(3-x)\sqrt[3]{(3-x)^2} - \frac{15}{2}\sqrt[3]{(3-x)^2} + C.$

103. $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}.$ Answer: $2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln(\sqrt[6]{x}+1) + C.$

104. $\int \frac{\sqrt{x-1}}{\sqrt{x-1}+1}dx.$ Answer: $x - 2\sqrt{x-1} + 2\ln(\sqrt{x-1}+1) + C.$

105. $\int \sqrt{\frac{1-x}{1+x}}\cdot\frac{dx}{x}.$ Answer: $\ln\left|\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right| + 2\arctan\sqrt{\frac{1-x}{1+x}} + C.$

Integration of the irrational functions. Trigonometric substitutions

106. $\int \sqrt{2-x^2}dx.$ Answer: $\arcsin\frac{x}{\sqrt{2}} + \frac{1}{2}x\sqrt{2-x^2} + C.$

107. $\int x^2\sqrt{9-x^2}dx$. Answer: $\frac{81}{8}\arcsin\frac{x}{3}-\frac{1}{8}x(9-2x^2)\sqrt{9-x^2}+C$.

108. $\int\sqrt{3-2x-x^2}dx$. Answer: $2\arcsin\frac{x+1}{2}+\frac{1}{2}(x+1)\sqrt{3-2x-x^2}+C$.

109. $\int\frac{x^2}{\sqrt{1-x^2}}dx$. Answer: $\frac{1}{2}\arcsin x-\frac{1}{2}x\sqrt{1-x^2}+C$.

110. $\int\frac{\sqrt{4+x^2}}{x^4}dx$. Answer: $-\frac{(4+x^2)\sqrt{4+x^2}}{12x^3}+C$.

111. $\int\frac{dx}{x^2\sqrt{x^2+16}}$. Answer: $-\frac{\sqrt{x^2+16}}{16x}+C$.

112. $\int\sqrt{4x^2+4x+5}dx$. Answer: $\ln\left(\sqrt{4x^2+4x+5}+2x+1\right)+\frac{2x+1}{4}\sqrt{4x^2+4x+5}+C$.

113. $\int\sqrt{x^2-3}dx$. Answer: $\frac{1}{2}x\sqrt{x^2-3}+\frac{3}{2}\ln\left|x-\sqrt{x^2-3}\right|+C$.

114. $\int\frac{\sqrt{x^2-4}}{x^4}dx$. Answer: $\frac{(x^2-4)\sqrt{x^2-4}}{12x^3}+C$.

115. $\int\sqrt{9x^2-6x}dx$. Answer: $\frac{1}{6}\ln\left|3x-1-\sqrt{9x^2-6x}\right|+\frac{1}{6}(3x-1)\sqrt{9x^2-6x}+C$.